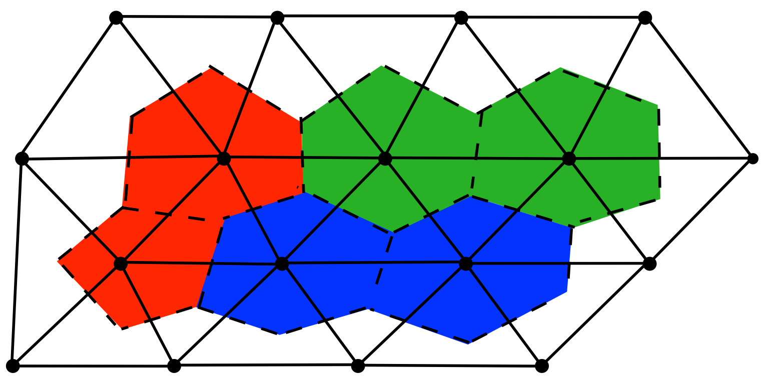
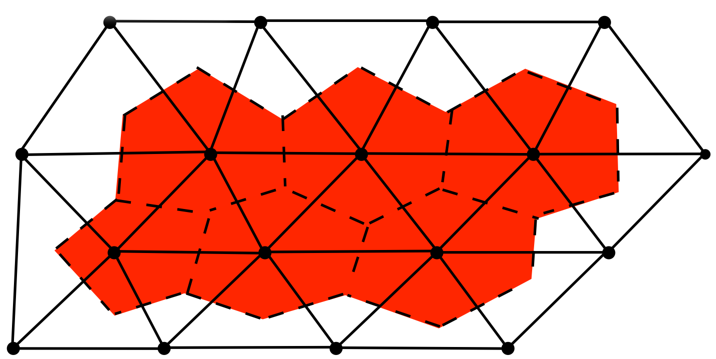
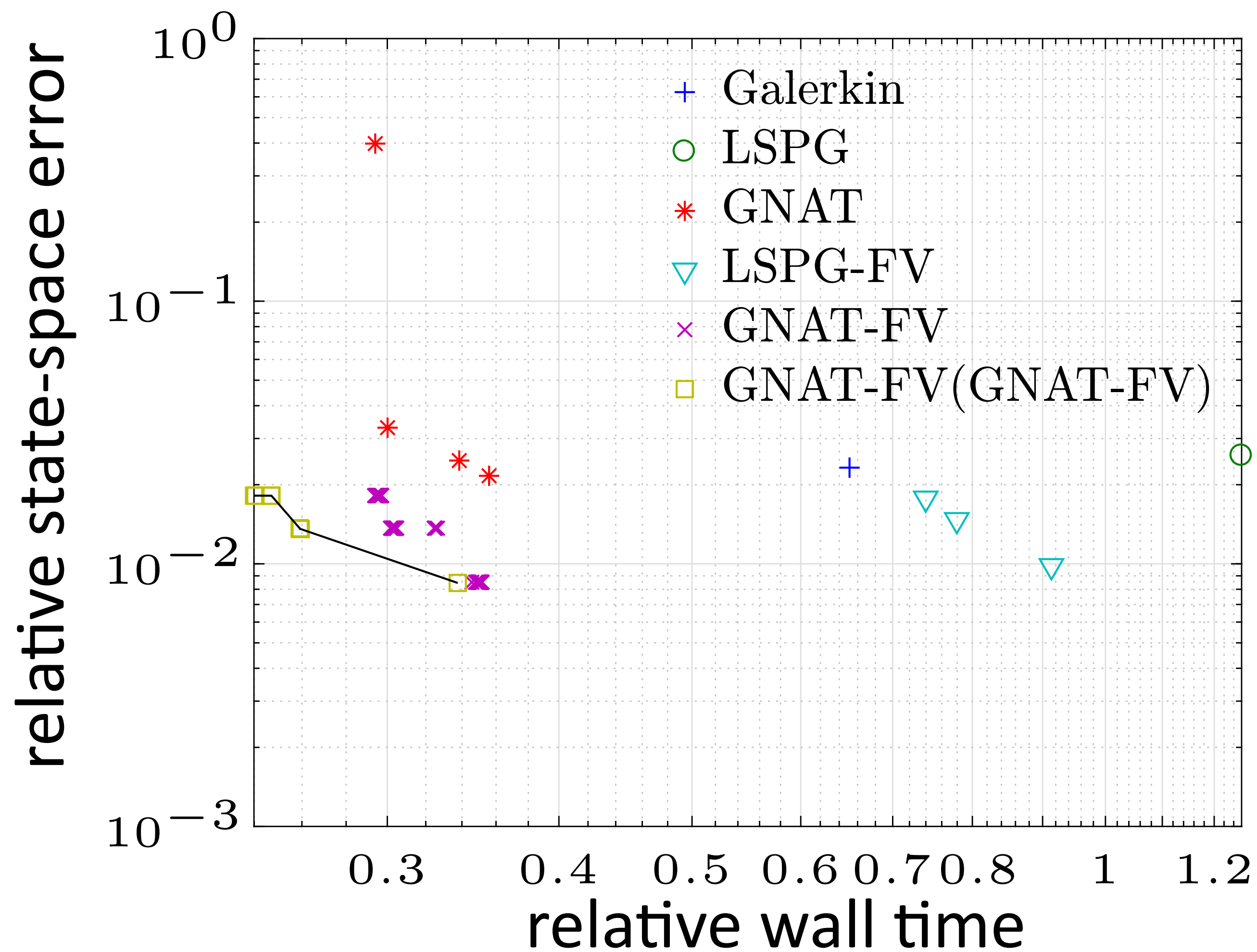
Conservative model reduction for finite-volume models in CFD



3 subdomains



1 (global) subdomain



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Sandia National Laboratories

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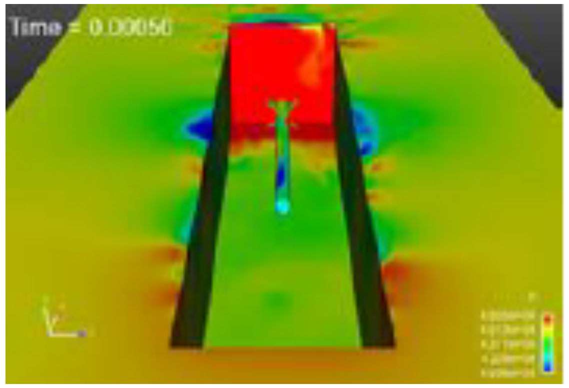
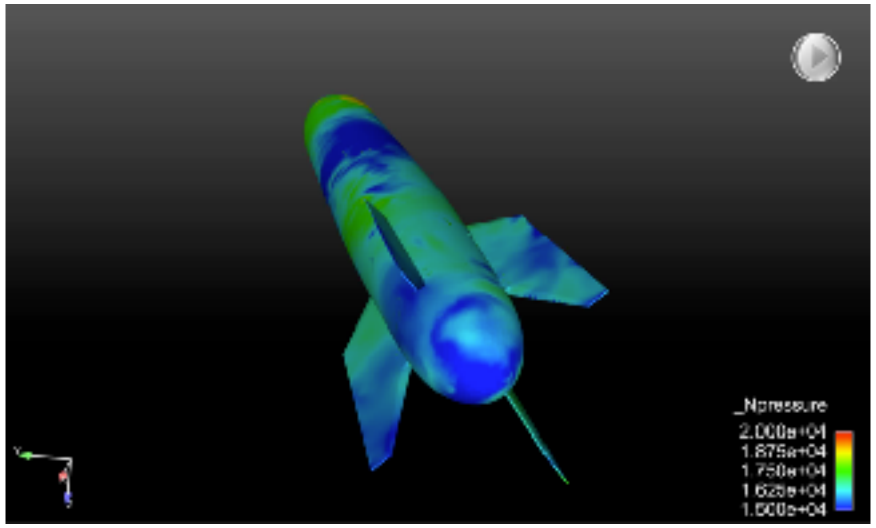
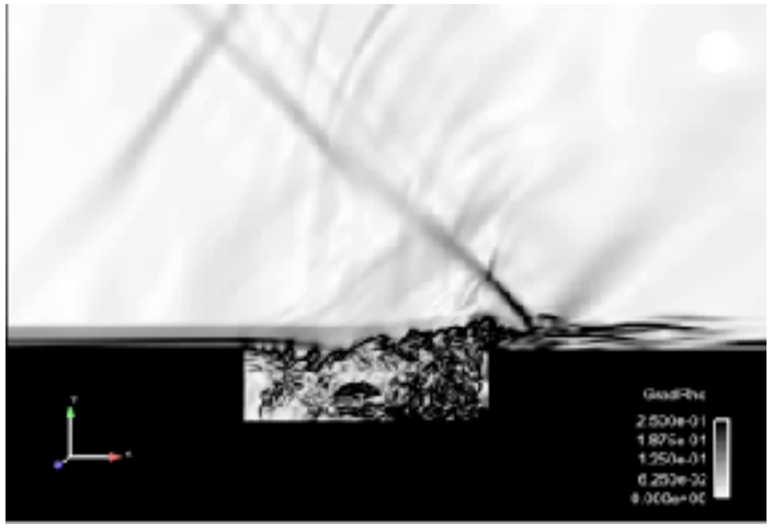
High-fidelity simulation: captive carry



High-fidelity simulation: captive carry





+ *Validated and predictive*: matches wind-tunnel experiments to within 5%
- *Extreme-scale*: 100 million cells, 200,000 time steps
- *High simulation costs*: 6 weeks, 5000 cores

computational barrier

Many-query problems

- explore flight envelope
- quantify effects of uncertainties on store load
- robust design of store and cavity

Goal: break computational barrier

How to construct a ROM given a basis $\boldsymbol{\Phi}$?

ODE

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; t)$$

residual minimization →

Galerkin ODE

$$\frac{d\hat{\mathbf{x}}}{dt} = \boldsymbol{\Phi}^T\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t)$$

↕

$$\mathbf{r}\left(\frac{d\mathbf{x}}{dt}, \mathbf{x}, t\right) = \mathbf{0}$$

$$\boldsymbol{\Phi}\frac{d\hat{\mathbf{x}}}{dt}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \underset{\mathbf{v}\in \mathrm{range}(\boldsymbol{\Phi})}{\arg\min} \|\mathbf{r}(\mathbf{v}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

time discretization ↓

LSPG OΔE

[C., Bou-Mosleh, Farhat, 2011]

$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n)^T\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) = \mathbf{0}$$
$$n = 1, \ldots, T$$

↕

$$\boldsymbol{\Phi}\hat{\mathbf{x}}^n = \underset{\mathbf{v}\in \mathrm{range}(\boldsymbol{\Phi})}{\arg\min} \|\mathbf{r}^n(\mathbf{v})\|_2$$
$$n = 1, \ldots, T$$

← *residual minimization*

OΔE

$$\mathbf{r}^n(\mathbf{x}^n) = 0$$
$$n = 1, \ldots, T$$

Galerkin OΔE

$$\boldsymbol{\Phi}^T\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) = 0$$
$$n = 1, \ldots, T$$

- FOM ODE residual: $\mathbf{r}(\mathbf{v}, \mathbf{x}, t) := \mathbf{v} - \mathbf{f}(\mathbf{x}, t)$
- FOM OΔE residual: $\mathbf{r}^n(\mathbf{w}) := \alpha_0\mathbf{w} - \Delta t\beta_0\mathbf{f}(\mathbf{w}, t^n) + \sum_{j=1}^{k}\alpha_j\mathbf{x}^{n-j}(\boldsymbol{\nu}) - \Delta t\sum_{j=1}^{k}\beta_j\mathbf{f}(\mathbf{x}^{n-j}, t^{n-j})$
- LSPG test basis: $\boldsymbol{\Psi}^n(\hat{\mathbf{w}}) := \left(\alpha_0\mathbf{I} + \beta_0\Delta t\frac{\partial\mathbf{f}}{\partial\mathbf{x}}(\boldsymbol{\Phi}\hat{\mathbf{w}}, t^n)\right)\boldsymbol{\Phi}$
- Detailed comparative analysis: C, Barone, Antil, *J Comp Phys*, 2017.

Discrete-time error bound

Theorem [C., Barone, Antil, 2017]

If the following conditions hold:

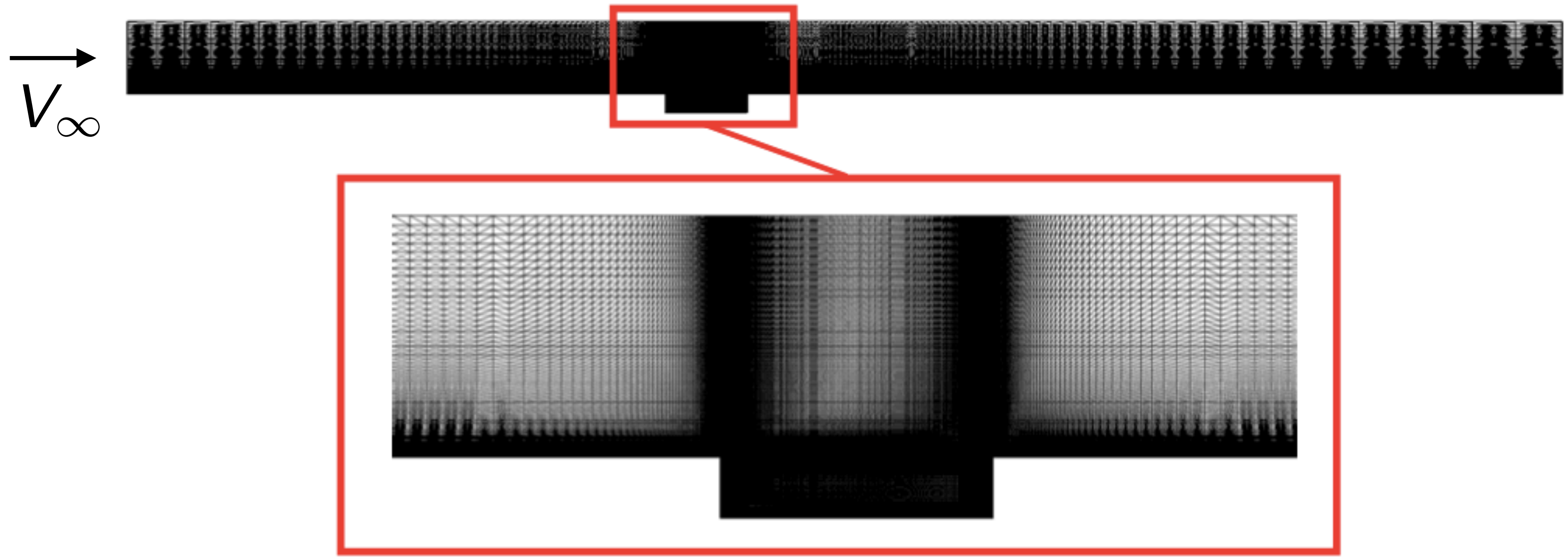
1. $\mathbf{f}(\cdot; t)$ is Lipschitz continuous with Lipschitz constant κ
2. The time step Δt is small enough such that $0 < h := |\alpha_0| - |\beta_0|\kappa\Delta t$,
3. A backward differentiation formula (BDF) time integrator is used,

$$\|\mathbf{x}^n - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^n\|_2 \leq \frac{1}{h}\|\mathbf{r}_{\mathrm{G}}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^n)\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^{n-\ell}\|_2$$

$$\|\mathbf{x}^n - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^n\|_2 \leq \frac{1}{h}\min_{\hat{\mathbf{v}}}\|\mathbf{r}_{\mathrm{LSPG}}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^{n-\ell}\|_2$$

\+ LSPG sequentially minimizes the error bound

B61 captive carry



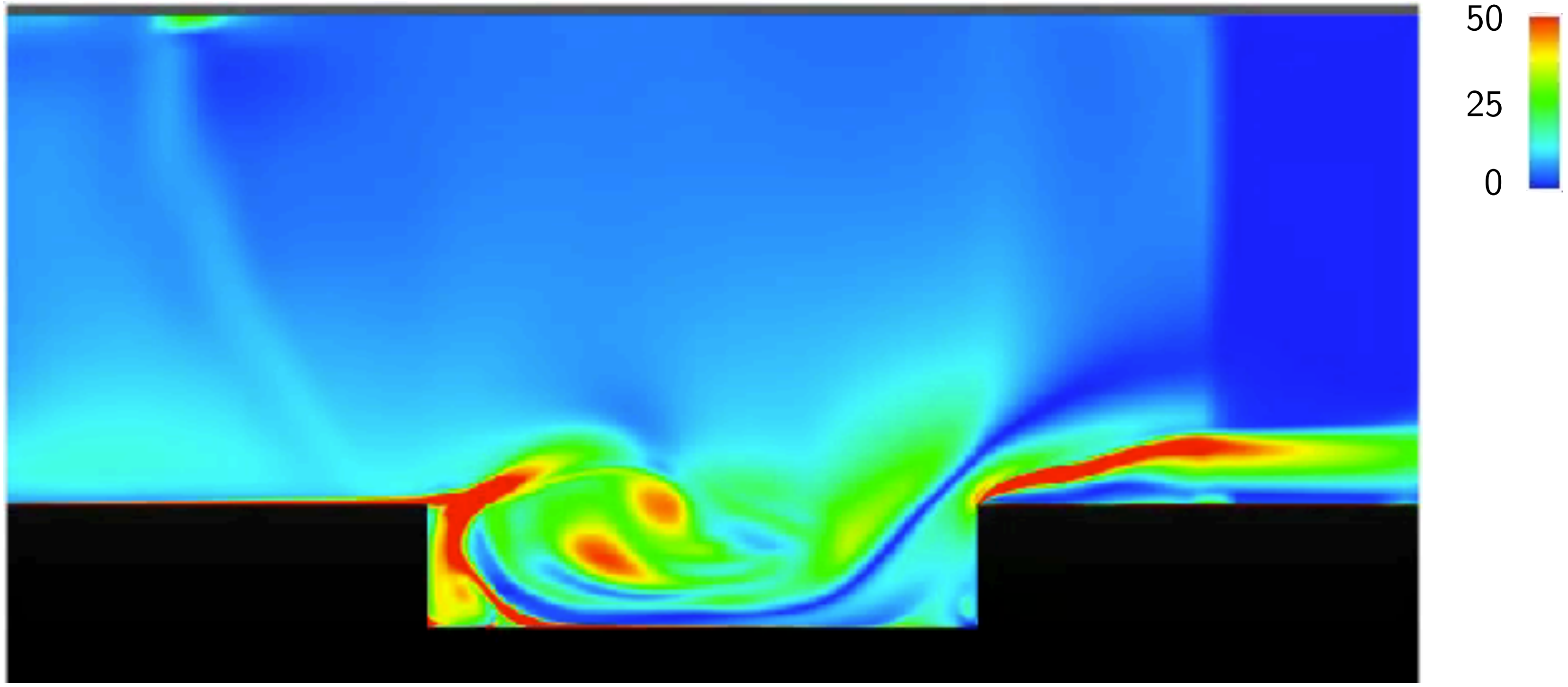
- Unsteady Navier–Stokes
- Re = 6.3 x 10^6
- $M_\infty = 0.6$

Spatial discretization
- 2nd-order finite volume
- DES turbulence model
- 1.2×10^6 degrees of freedom

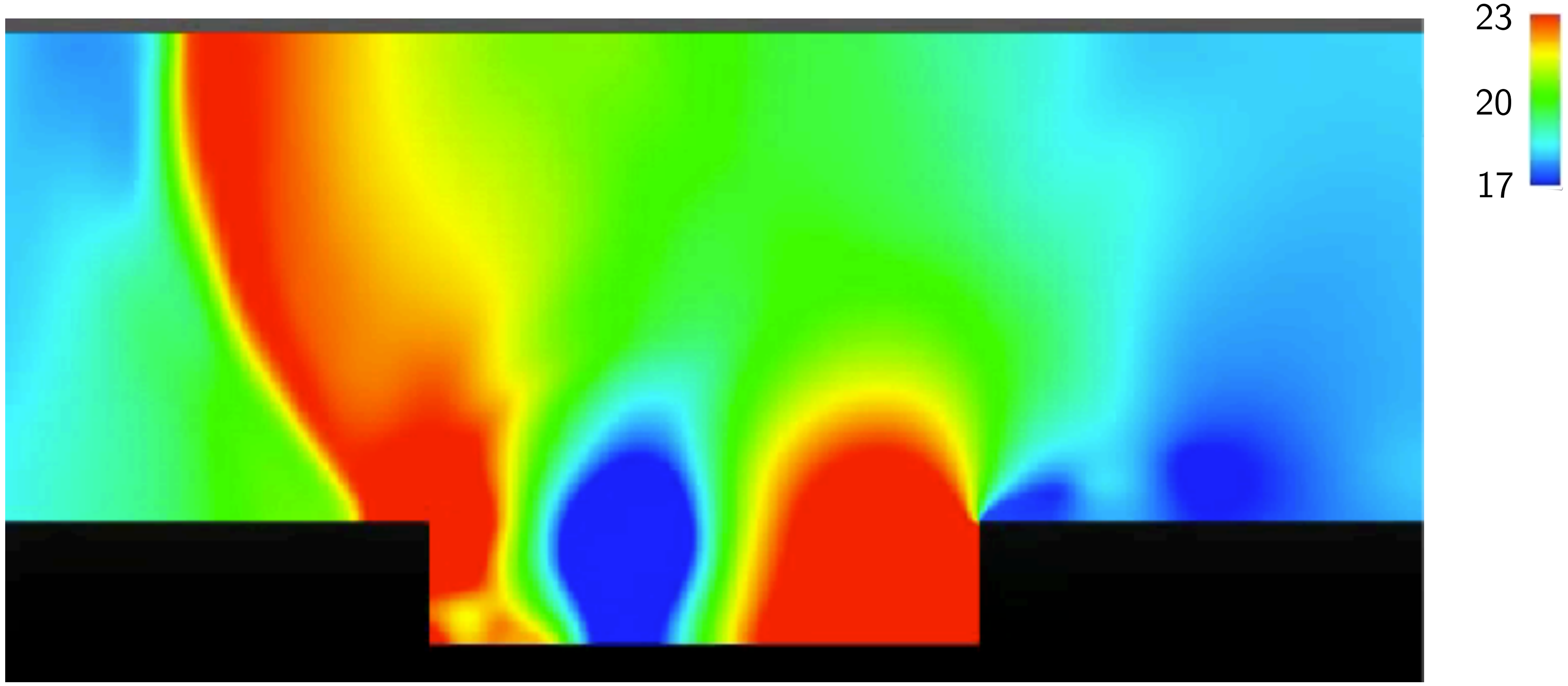
Temporal discretization
- 2nd-order BDF
- Verified time step $\Delta t = 1.5 \times 10^{-3}$
- 8.3×10^3 time instances

High-fidelity model solution

vorticity field

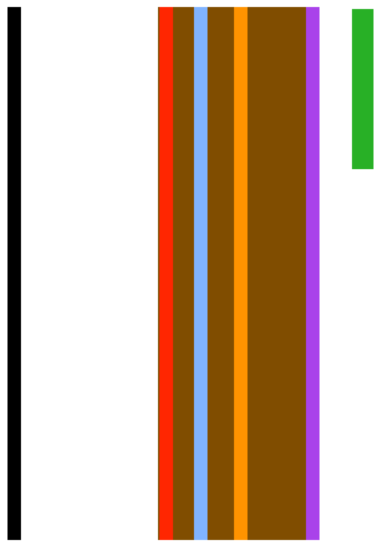


pressure field

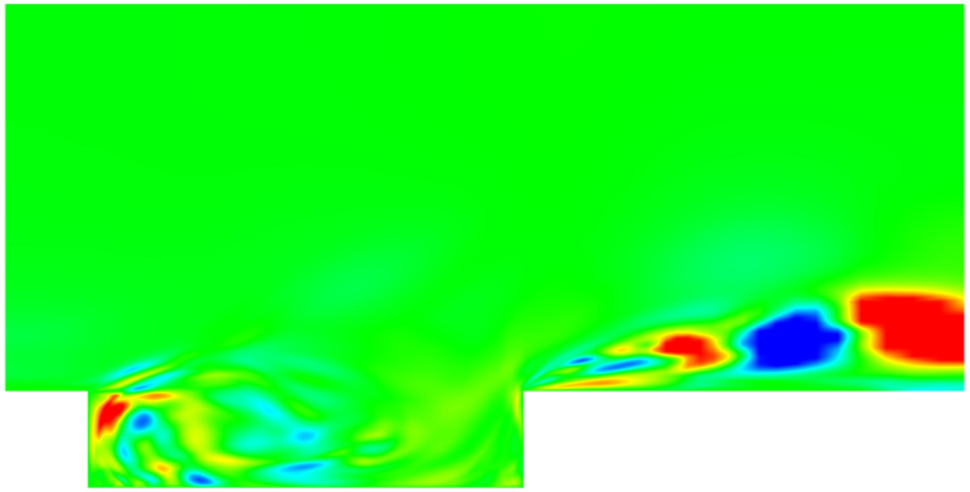


Principal components

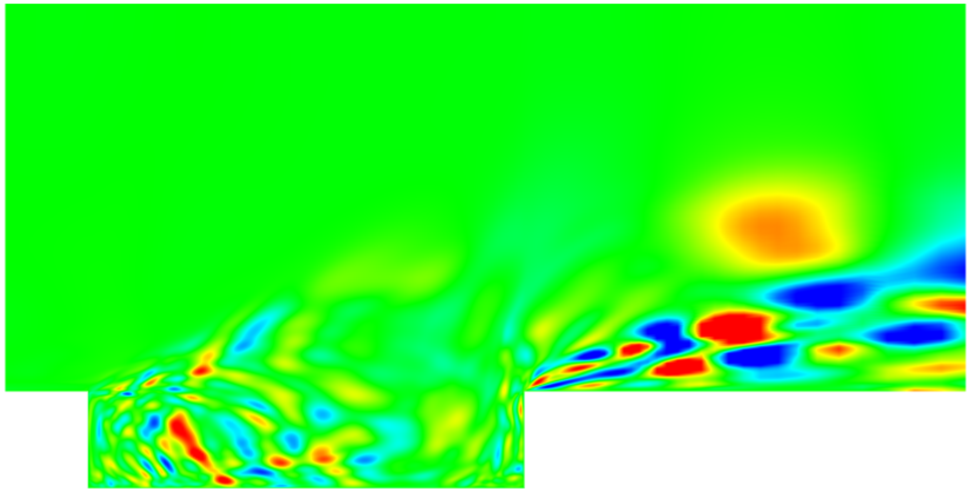
$\mathbf{x}(t) \approx \mathbf{\Phi}\, \hat{\mathbf{x}}(t)$



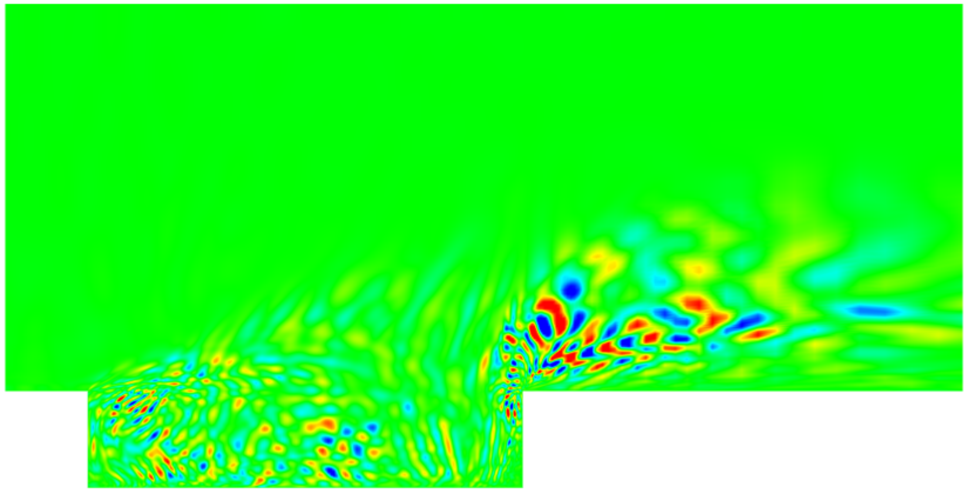
ϕ_1



ϕ_{21}

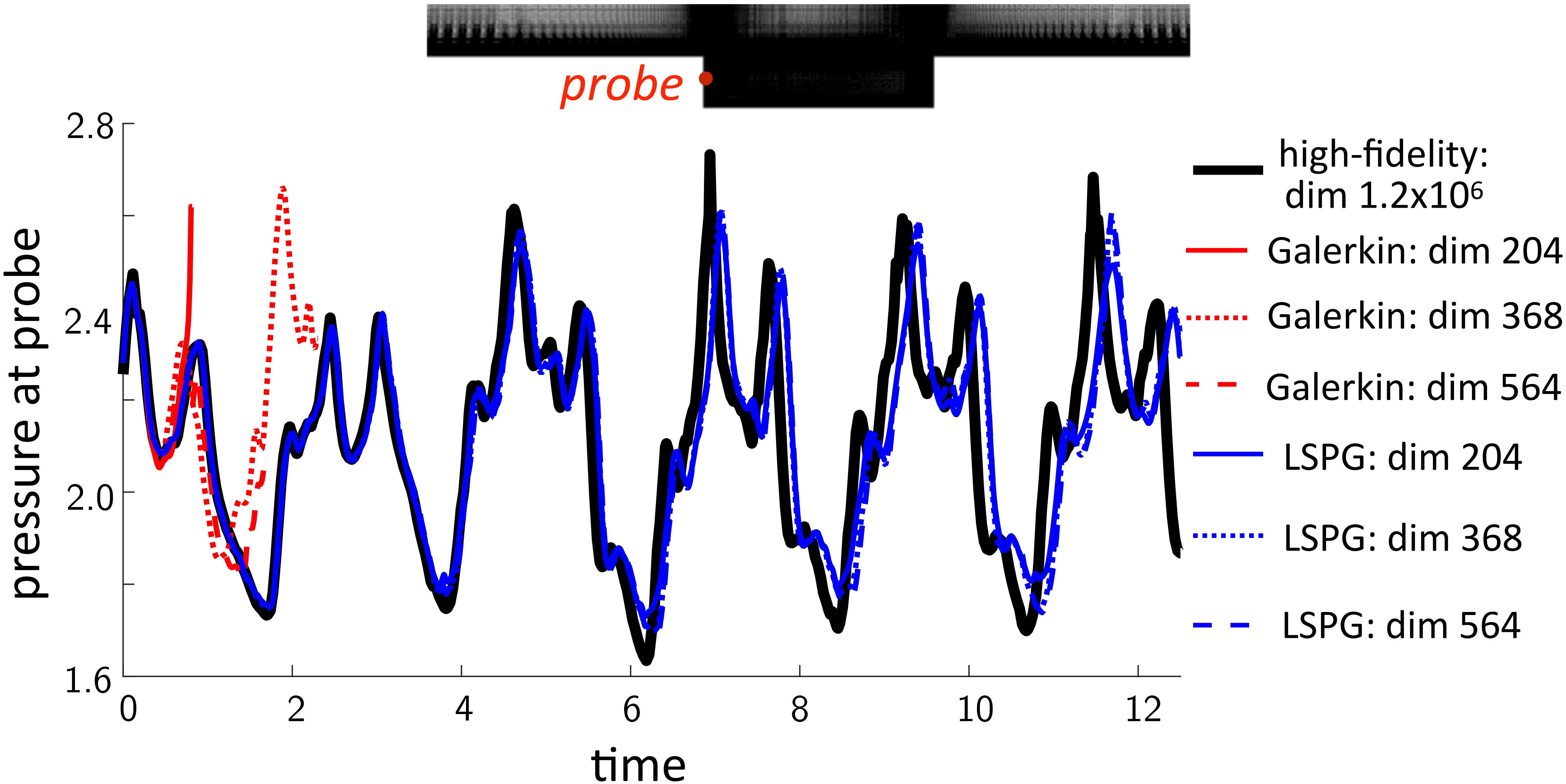


ϕ_{101}



ϕ_{401}

Galerkin and LSPG performance



- Galerkin projection fails regardless of basis dimension

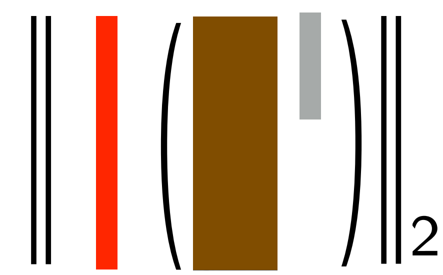
+ LSPG is far more accurate than Galerkin

- However, both ROMs are slower than the high-fidelity model

Why does this occur, and can we fix it?

Hyper-reduction

Galerkin: $\underset{\hat{\mathbf{v}}}{\text{minimize}} \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$ **LSPG:** $\underset{\hat{\mathbf{v}}}{\text{minimize}} \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$

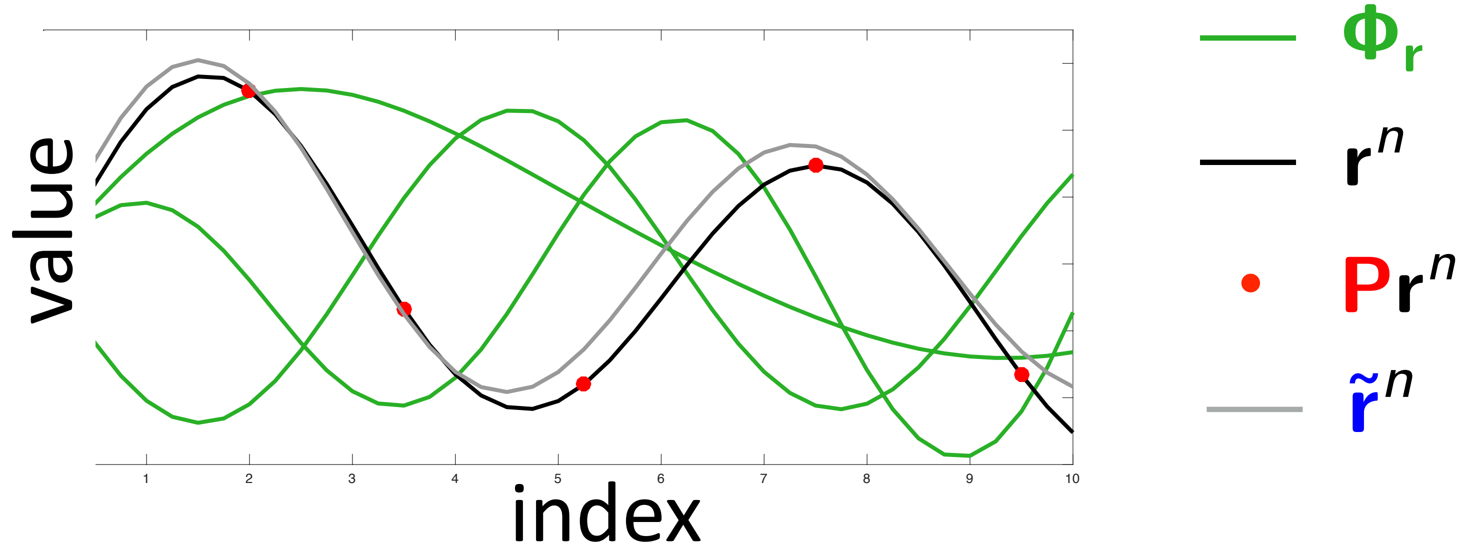


- Costly: minimizing large-scale high-fidelity model residual

Hyper-reduction: minimize sampling-based residual approximations

HR-Galerkin: $\underset{\hat{\mathbf{v}}}{\text{minimize}} \|\tilde{\mathbf{r}}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$ **HR-LSPG:** $\underset{\hat{\mathbf{v}}}{\text{minimize}} \|\tilde{\mathbf{r}}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$

1. Residual gappy POD: $\tilde{\mathbf{r}} = \boldsymbol{\Phi}_\mathbf{r}(\mathbf{P}_\mathbf{r}\boldsymbol{\Phi}_\mathbf{r})^+\mathbf{P}_\mathbf{r}\mathbf{r}$, $\tilde{\mathbf{r}}^n = \boldsymbol{\Phi}_\mathbf{r}(\mathbf{P}_\mathbf{r}\boldsymbol{\Phi}_\mathbf{r})^+\mathbf{P}_\mathbf{r}\mathbf{r}^n$



+ Cost independent of high-fidelity model dimension

- GNAT [C., Bou-Mosleh, Farhat, 2011] = LSPG + residual gappy POD

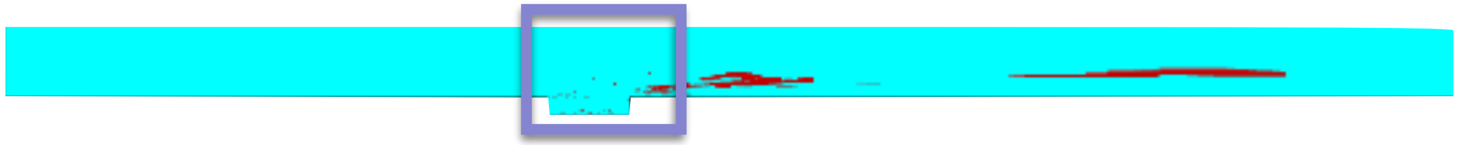
2. Velocity gappy POD: $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{r}}^n$ computed from $\tilde{\mathbf{f}} = \boldsymbol{\Phi}_\mathbf{f}(\mathbf{P}_\mathbf{f}\boldsymbol{\Phi}_\mathbf{f})^+\mathbf{P}_\mathbf{f}\mathbf{f}$

- POD-DEIM [Chaturantabut and Sorensen, 2011] = Galerkin + velocity gappy POD

Sample mesh [C., Farhat, Cortial, Amsallem, 2013]

$$\underset{\hat{\mathbf{v}}}{\text{minimize}} \|(\mathbf{P}\boldsymbol{\Phi}_{\mathbf{r}})^{+}\underbrace{\mathbf{P}\mathbf{r}^{n}}(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

sample mesh

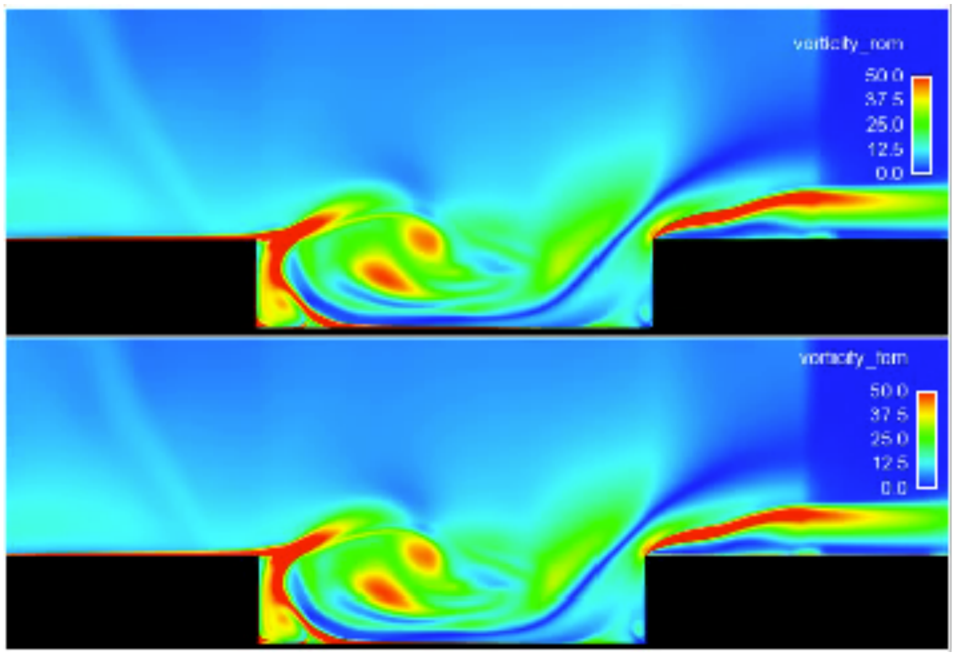


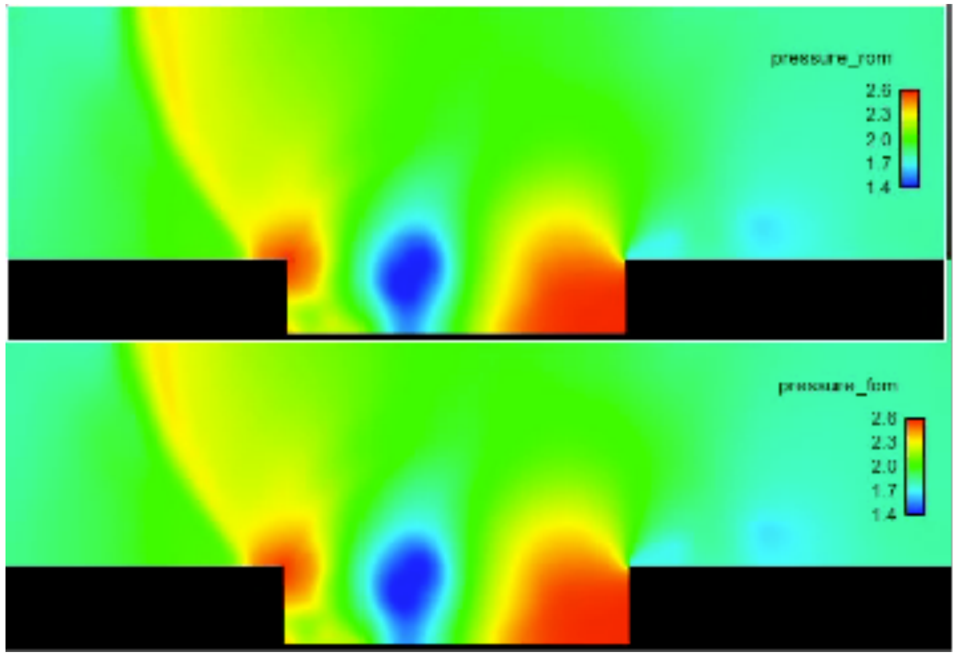
+ *HPC on a laptop*

vorticity field | *pressure field*

GNAT ROM
32 min, 2 cores

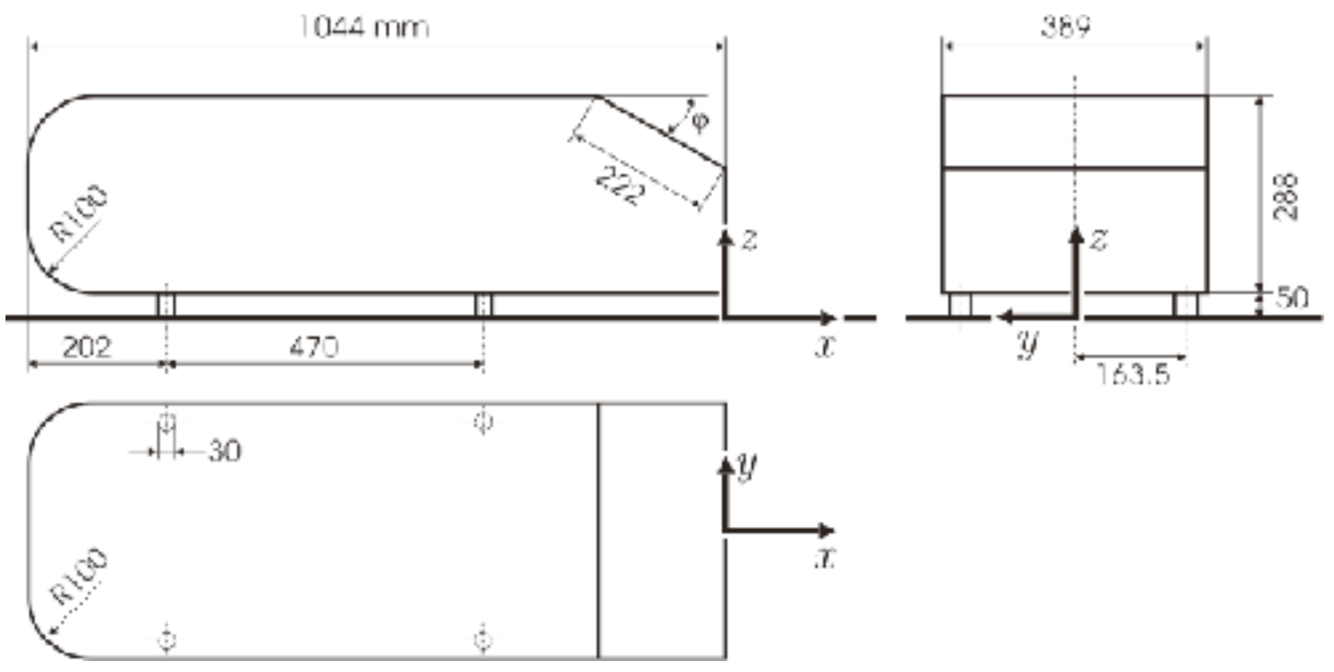
high-fidelity
5 hours, 48 cores

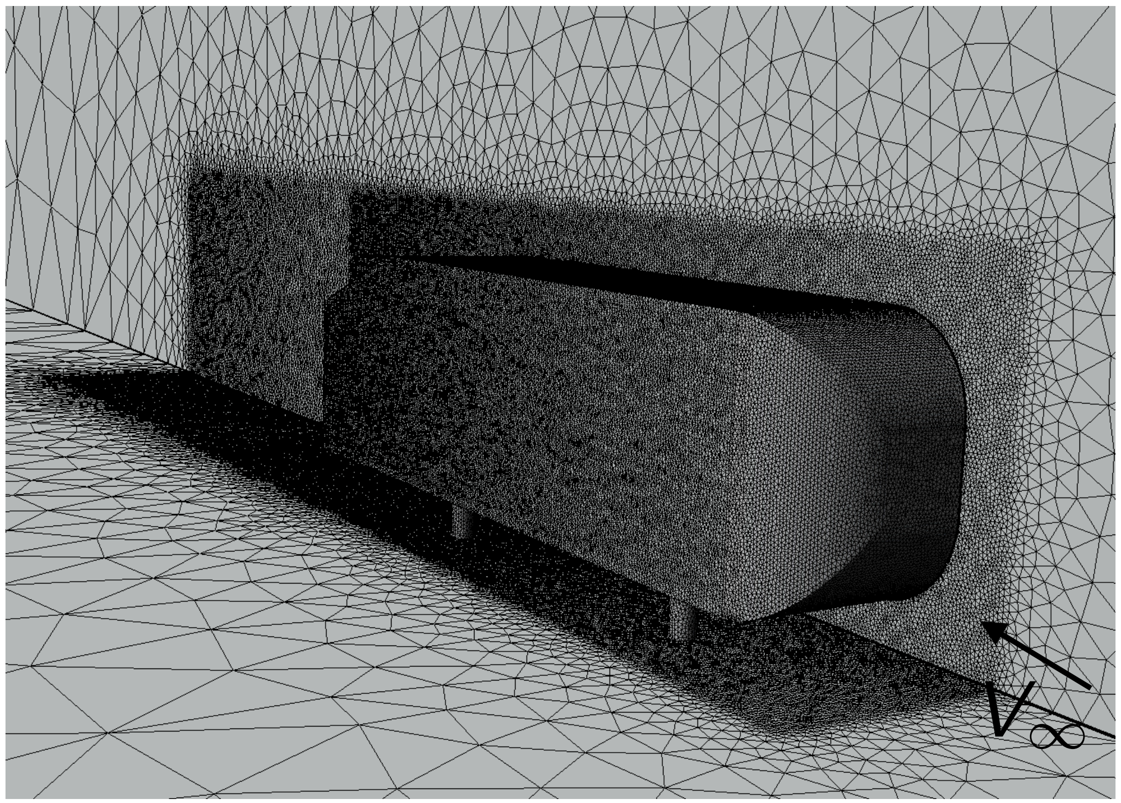




+ *229x savings in core–hours*
+ *< 1% error in time-averaged drag*

Ahmed body [Ahmed, Ramm, Faitin, 1984]





- Unsteady Navier–Stokes
- Re = 4.3 x 10^6
- M_∞ = 0.175

Spatial discretization

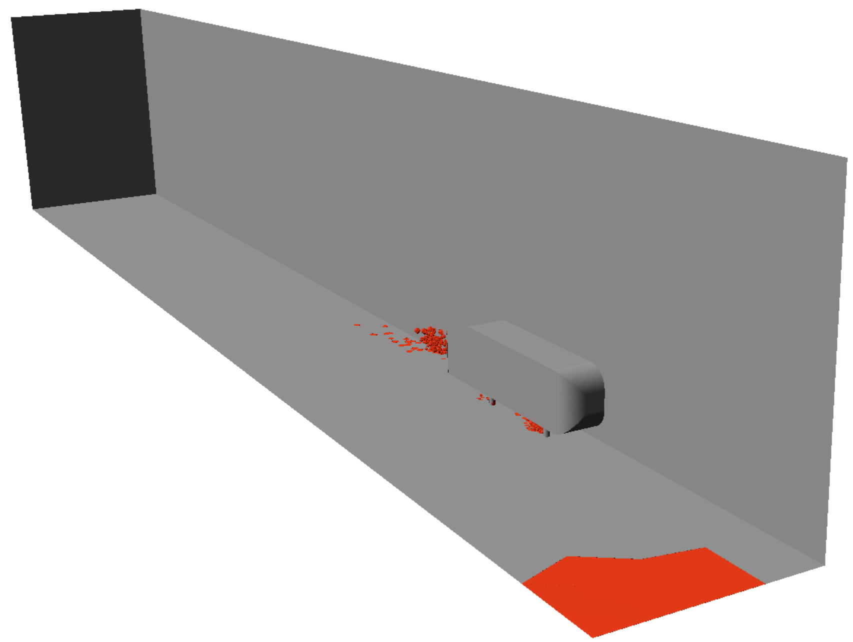
- 2nd-order finite volume
- DES turbulence model
- 1.7×10^7 degrees of freedom

Temporal discretization

- 2nd-order BDF
- Time step $\Delta t = 8 \times 10^{-5}\mathrm{s}$
- 1.3×10^3 time instances

Ahmed body results [C., Farhat, Cortial, Amsallem, 2013]

sample mesh



+ *HPC on a laptop*

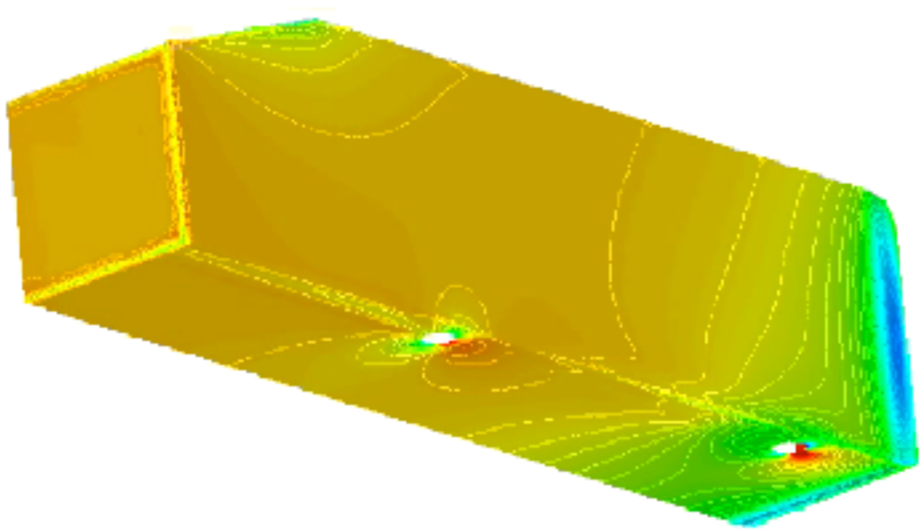
GNAT ROM

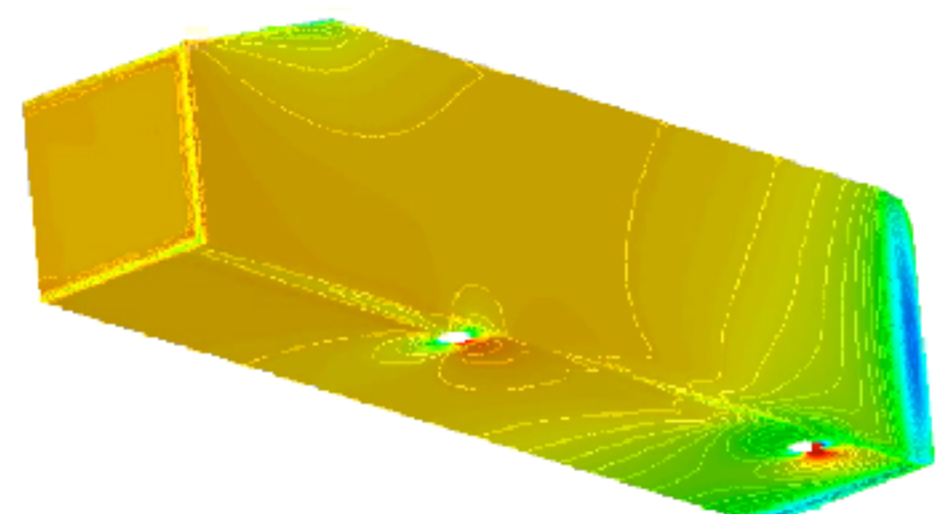
4 hours, 4 cores

high-fidelity model

13 hours, 512 cores

pressure field





+ *438x savings in core–hours*

Can we equip the ROM with stronger a priori guarantees?

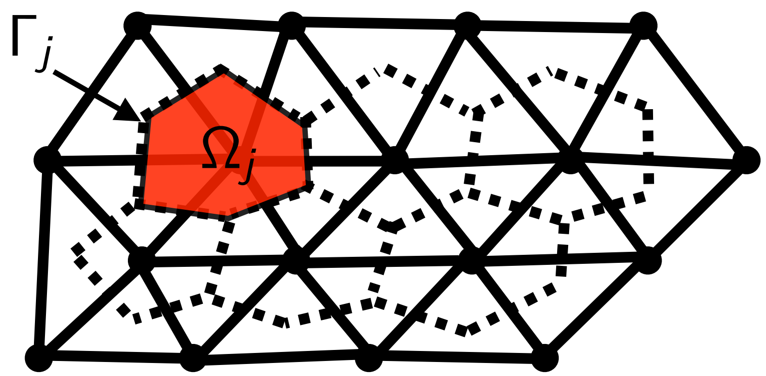
Structure preservation in model reduction

- Stability [Moore, 1981; Bond and Daniel, 20018; Amsallem and Farhat, 2012; Kalashnikova et al., 2014]
- Second-order structure [Freund 2005; Salimbahrami, 2005; Chahlaoui, 2015]
- Delay [Beattie and Gugercin, 2008; Michiels et al., 2011; Schulze and Unger, 2015]
- Bilinear [Zhang and Lam, 2002; Benner and Damm, 2011; Benner and Breiten, 2012; Flagg and Gugercin, 2015]
- Inf–sup stability [Rozza and Veroy, 2007; Gerner and Veroy, 2012; Rozza et al., 2013; Ballarin et al., 2014]
- Passivity [Phillips et al., 2003; Sorensen 2005; Wolf et al., 2010]
- Energy conservation [Farhat et al., 2014; Farhat et al., 2015]
- (Port-)Hamiltonian [Polyuga and van der Schaft, 2008; Beattie and Gugercin, 2011; Arkham and Hesthaven, 2016; Chaturantabut et al., 2016; Peng and Mohseni, 2016]

What structure should we preserve in finite-volume models?

Finite-volume method

ODE: $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; t)$



$$x_{\mathcal{I}(i,j)}(t) = \frac{1}{|\Omega_j|} \int_{\Omega_j} u_i(\vec{x}, t)\, d\vec{x}$$

- average value of conserved variable i over control volume j

$$f_{\mathcal{I}(i,j)}(\mathbf{x}, t) = -\frac{1}{|\Omega_j|} \int_{\Gamma_j} \underbrace{\mathbf{g}_i(\mathbf{x}; \vec{x}, t)}_{\text{flux}} \cdot \mathbf{n}_j(\vec{x})\, d\vec{s}(\vec{x}) + \frac{1}{|\Omega_j|} \int_{\Omega_j} \underbrace{s_i(\mathbf{x}; \vec{x}, t)}_{\text{source}}\, d\vec{x}$$

- flux and source of conserved variable i within control volume j

$$r_{\mathcal{I}(i,j)} = \frac{dx_{\mathcal{I}(i,j)}}{dt}(t) - f_{\mathcal{I}(i,j)}(\mathbf{x}, t)$$

- **rate of conservation violation** of variable i in control volume j

OΔE: $\mathbf{r}^n(\mathbf{x}^n) = 0, \quad n = 1, \ldots, N$

$$r^n_{\mathcal{I}(i,j)} = x_{\mathcal{I}(i,j)}(t^{n+1}) - x_{\mathcal{I}(i,j)}(t^n) + \int_{t^n}^{t^{n+1}} f_{\mathcal{I}(i,j)}(\mathbf{x}, t) dt$$

- **conservation violation** of variable i in control volume j over time step n

Conservation is the intrinsic structure enforced by finite-volume methods

Galerkin and LSPG violate conservation

Galerkin

$$\Phi \frac{d\hat{\mathbf{x}}}{dt}(\Phi\hat{\mathbf{x}}, t) = \underset{\mathbf{v}\in\text{range}(\Phi)}{\arg\min} \|\mathbf{r}(\mathbf{v}, \Phi\hat{\mathbf{x}}, t)\|_2$$

- Minimize sum of squared **conservation-violation rates** over all conserved variables and control volumes

LSPG

$$\Phi\hat{\mathbf{x}}^n = \underset{\mathbf{v}\in\text{range}(\Phi)}{\arg\min} \|\mathbf{r}^n(\mathbf{v})\|_2$$

- Minimize sum of squared **conservation violations over time step *n*** over all conserved variables and control volumes

- Neither Galerkin nor LSPG enforces conservation!

Objectives

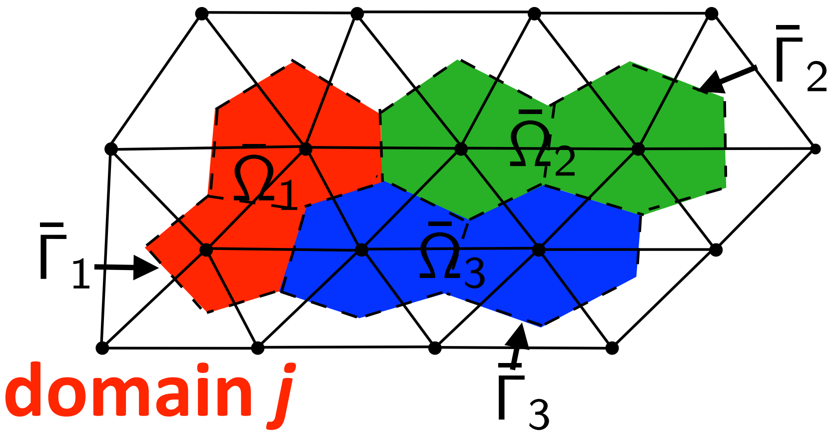
+ Reduced-order models that enforce conservation
+ Conditions that determine when conservation enforcement is ensured
+ Hyper-reduction to ensure low cost if nonlinear flux and source
+ *A posteriori* error bounds

Approach*: leverage optimization structure of Galerkin and LSPG*

Reference: C., Choi, and Sargsyan. Conservative model reduction for finite-volume models. *Journal of Computational Physics*, 371:280–314, 2018.

Finite-volume method **over subdomains**

$$\text{ODE: } \bar{\mathbf{C}}\frac{d\mathbf{x}}{dt} = \bar{\mathbf{C}}\mathbf{f}(\mathbf{x}, t)$$



$$\bar{c}_{\bar{\mathcal{I}}(i,j),\mathcal{I}(\ell,k)} = |\Omega_k|/|\bar{\Omega}_j|\delta_{i\ell} I(\Omega_k \subseteq \bar{\Omega}_j)$$

- performs summation over control volumes within **subdomain *j***

$$[\bar{\mathbf{C}}\mathbf{x}(t)]_{\bar{\mathcal{I}}(i,j)}(\mathbf{x}, t; \boldsymbol{\mu}) = \frac{1}{|\bar{\Omega}_j|}\int_{\bar{\Omega}_j} u_i(\vec{x}, t; \boldsymbol{\mu})\, \mathrm{d}\vec{x}$$

- average value of conserved variable *i* over **subdomain *j***

$$[\bar{\mathbf{C}}\mathbf{f}(\mathbf{x}, t)]_{\bar{\mathcal{I}}(i,j)} = -\frac{1}{|\bar{\Omega}_j|}\int_{\bar{\Gamma}_j} \underbrace{\mathbf{g}_i(\mathbf{x}; \vec{x}, t)}_{\text{flux}} \cdot \bar{\mathbf{n}}_j(\vec{x})\, \mathrm{d}\vec{s}(\vec{x}) + \frac{1}{|\bar{\Omega}_j|}\int_{\bar{\Omega}_j} \underbrace{s_i(\mathbf{x}; \vec{x}, t)}_{\text{source}}\, \mathrm{d}\vec{x}$$

- flux and source of conserved variable *i* within **subdomain *j***

$$[\bar{\mathbf{C}}\mathbf{r}]_{\bar{\mathcal{I}}(i,j)} = d[\bar{\mathbf{C}}\mathbf{x}(t)]_{\bar{\mathcal{I}}(i,j)}/dt - [\bar{\mathbf{C}}\mathbf{f}(\mathbf{x}, t)]_{\bar{\mathcal{I}}(i,j)}$$

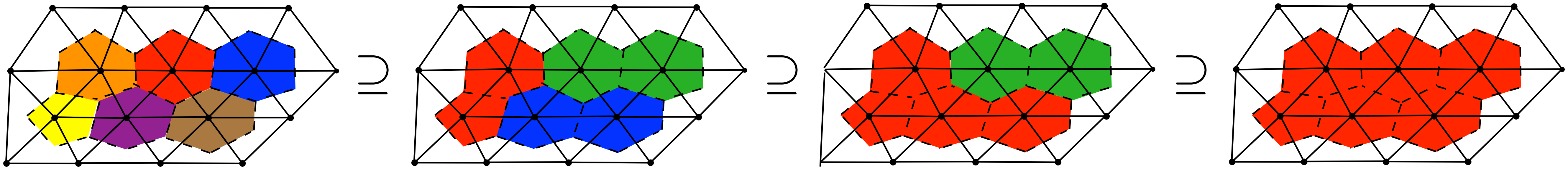
- **rate of conservation violation** of conserved variable *i* in **subdomain *j***

$$\text{O}\Delta\text{E: } \bar{\mathbf{C}}\mathbf{r}^n(\mathbf{x}^n) = \mathbf{0}, \quad n = 1, \ldots, T$$

$$[\bar{\mathbf{C}}\mathbf{r}^n]_{\bar{\mathcal{I}}(i,j)} = [\bar{\mathbf{C}}\mathbf{x}(t^{n+1})]_{\bar{\mathcal{I}}(i,j)} - [\bar{\mathbf{C}}\mathbf{x}(t^n)]_{\bar{\mathcal{I}}(i,j)} + \int_{t^n}^{t^{n+1}} [\bar{\mathbf{C}}\mathbf{f}(\mathbf{x}, t)]_{\bar{\mathcal{I}}(i,j)} dt$$

- **conservation violation** of conserved variable *i* in **subdomain *j*** over time step *n*

Nested conservation



$\mathcal{M} \supseteq \bar{\mathcal{M}}_1 \supseteq \bar{\mathcal{M}}_2 \supseteq \bar{\mathcal{M}}_{\mathrm{global}}$

Theorem: Nested conservation [C., Choi, Sargsyan, 2018]

- If a decomposed mesh $\bar{\bar{\mathcal{M}}}$ is nested in another decomposed mesh $\bar{\mathcal{M}}$ such that $\bar{\bar{\Omega}}_i = \cup_{j \in \bar{\mathcal{K}} \subseteq \{1,\ldots,N_{\bar{\Omega}}\}} \bar{\Omega}_j$, $i = 1, \ldots, N_{\bar{\bar{\Omega}}}$, then we say $\bar{\bar{\mathcal{M}}} \subseteq \bar{\mathcal{M}}$.
- If $\bar{\bar{\mathcal{M}}} \subseteq \bar{\mathcal{M}}$ and $\bar{\mathcal{M}}$ is non-overlapping, then satisfaction of conservation on $\bar{\mathcal{M}}$ implies satisfaction of conservation on $\bar{\bar{\mathcal{M}}}$, i.e.,

$$\bar{\mathbf{C}}\mathbf{r}\left(\frac{d\mathbf{x}}{dt}, \mathbf{x}, t\right) = \mathbf{0} \Rightarrow \bar{\bar{\mathbf{C}}}\mathbf{r}\left(\frac{d\mathbf{x}}{dt}, \mathbf{x}, t\right) = \mathbf{0}, \qquad \bar{\mathbf{C}}\mathbf{r}^n(\mathbf{x}^n) = 0 \Rightarrow \bar{\bar{\mathbf{C}}}\mathbf{r}^n(\mathbf{x}^n) = \mathbf{0}$$

Corollary: Global conservation [C., Choi, Sargsyan, 2018]

If the decomposed mesh $\bar{\mathcal{M}}$ satisfies $\cup_{i=1}^{N_{\bar{\Omega}}} \bar{\Omega}_i = \Omega$ and is non-overlapping, then it is globally conservative.

Conservative model reduction

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

- Minimize sum of squared **conservation-violation rates** over all conserved variables and control volumes subject to zero conservation-violation **rates** over subdomains

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- Minimize sum of squared **conservation violations over time step *n*** over all conserved variables and control volumes subject to zero conservation violations over time step *n* over subdomains

+ If feasible, ROMs enforce conservation over subdomains

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- How to handle infeasibility?
- How to solve?
- Are the two methods ever equivalent?
- How to apply hyper-reduction in a structure-preserving way?
- How do *a posteriori* error bounds compare with standard ROMs?

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- **What are conditions for feasibility?**
- How to handle infeasibility?
- How to solve?
- Are the two methods ever equivalent?
- How to apply hyper-reduction in a structure-preserving way?
- How do *a posteriori* error bounds compare with standard ROMs?

Conservative Galerkin feasibility

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Definition: conservative Galerkin feasibility

The conservative Galerkin model is feasible if the Galerkin feasible set

$$\mathcal{F}_{\mathrm{G}}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) := \{\hat{\mathbf{v}} \in \mathbb{R}^p \mid \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}\}$$

is non-empty.

Proposition: sufficient conditions for conservative Galerkin feasibility

The conservative Galerkin model is feasible, i.e., $\mathcal{F}_{\mathrm{G}}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) \neq \emptyset$ if $\bar{\mathbf{C}}\boldsymbol{\Phi}$ has full row rank (i.e., inf–sup stability). This in turn requires fewer constraints (i.e., rows in $\bar{\mathbf{C}}$) than unknowns (i.e., columns in $\boldsymbol{\Phi}$).

Constraint equations should be underdetermined.

Conservative LSPG feasibility

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

Definition: conservative LSPG feasibility

The conservative LSPG model is feasible if the LSPG feasible set

$$\mathcal{F}_{\mathrm{P}}^n := \{\hat{\mathbf{v}} \in \mathbb{R}^p \,|\, \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}\}$$

is non-empty.

Proposition: sufficient conditions for conservative LSPG feasibility

The conservative LSPG model is feasible, i.e., $\mathcal{F}_{\mathrm{P}}^n \neq \emptyset$ if

1. an explicit time integrator is used and $\bar{\mathbf{C}}\boldsymbol{\Phi}$ has full row rank
2. the limit $\Delta t \rightarrow 0$ is taken, or
3. The velocity $\mathbf{f}$ is linear in the state and $\bar{\mathbf{C}}[\alpha_0\mathbf{I} - \Delta t\beta_0\partial\mathbf{f}/\partial\mathbf{x}(\cdot, t^n)]\boldsymbol{\Phi}$ has full row rank.

Constraint equations should be underdetermined.

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

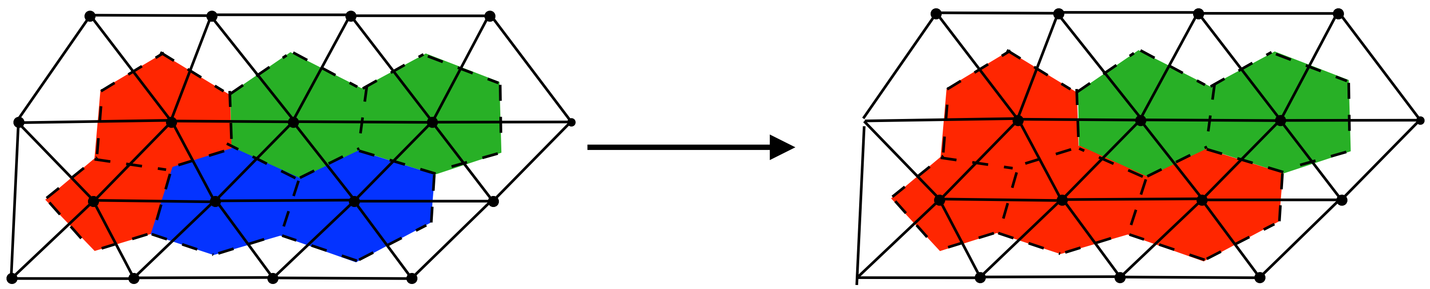
$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- **How to handle infeasibility?**
- How to solve?
- Are the two methods ever equivalent?
- How to apply hyper-reduction in a structure-preserving way?
- How do *a posteriori* error bounds compare with standard ROMs?

Handling infeasibility

What if infeasibility is detected?

1. Reduce number of subdomains



+ Fewer constraints, so likelihood of feasibility increases
+ Nested: solutions at previous time steps are feasible on new mesh
- No guarantee of feasibility (global conservation may be infeasible)

2. Penalty formulation

- Penalized Galerkin: $\underset{\hat{\mathbf{v}} \in \mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2^2 + \rho \|\bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2^2$
- Penalized LSPG: $\underset{\hat{\mathbf{v}} \in \mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2^2 + \rho \|\bar{\mathbf{C}}\mathbf{r}^n(\mathbf{x}^0(\boldsymbol{\mu}) + \boldsymbol{\Phi}\hat{\mathbf{v}})\|_2^2$

+ Always solvable
- No longer strictly conservative

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- How to handle infeasibility?
- **How to solve?**
- Are the two methods ever equivalent?
- How to apply hyper-reduction in a structure-preserving way?
- How do *a posteriori* error bounds compare with standard ROMs?

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Convex linear least-squares problem with linear equality constraints

Theorem

If the conservative Galerkin model is feasible, i.e., $\mathcal{F}_{\mathrm{G}}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) \neq \emptyset$ then its solution exists, is unique, and satisfies the following:

1. a time-dependent saddle point problem

$$\begin{bmatrix} \mathbf{I} & \boldsymbol{\Phi}^T\bar{\mathbf{C}}^T \\ \bar{\mathbf{C}}\boldsymbol{\Phi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \frac{d\hat{\mathbf{x}}}{dt} \\ \frac{d\boldsymbol{\lambda}_{\mathrm{G}}}{dt} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}^T\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) \\ \bar{\mathbf{C}}\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t; \boldsymbol{\mu}) \end{bmatrix}$$

2. a modified Galerkin projection

$$\frac{d\hat{\mathbf{x}}}{dt} = \boldsymbol{\Phi}^T\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) + \underbrace{(\bar{\mathbf{C}}\boldsymbol{\Phi})^+[\bar{\mathbf{C}}\mathbf{f}(\mathbf{x}, t; \boldsymbol{\nu}) - \bar{\mathbf{C}}\boldsymbol{\Phi}\boldsymbol{\Phi}^T\mathbf{f}(\mathbf{x}, t)]}_{\text{modification from Galerkin velocity}}$$

3. orthogonal projection of the Galerkin velocity onto the feasible set

$$\frac{d\hat{\mathbf{x}}}{dt}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \underset{\mathbf{v}\in\mathcal{F}_{\mathrm{G}}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t)}{\arg\min} \; \|\mathbf{v} - \boldsymbol{\Phi}^T\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

- **Solver**: any time integrator applied to these systems of ODEs

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}}\ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

Non-convex nonlinear least-squares problem with nonlinear equality constraints

Theorem

If the conservative LSPG model is feasible, i.e., $\mathcal{F}_{\mathrm{P}}^n \neq \emptyset$, then its solution exists and satisfies the nonlinear saddle-point problem

$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n)^T\left[\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) + \bar{\mathbf{C}}^T\boldsymbol{\lambda}_{\mathrm{P}}^n\right] = \mathbf{0}$$

$$\bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}^n) = \mathbf{0}$$

- **Solver**: SQP with Gauss–Newton Hessian approximation

$$\begin{bmatrix}\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^{n(k)})^T\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^{n(k)}) & \boldsymbol{\Psi}^n(\hat{\mathbf{x}}^{n(k)})^T\bar{\mathbf{C}}^T \\ \bar{\mathbf{C}}\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^{n(k)}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta\hat{\mathbf{x}}^{n(k)} \\ \delta\boldsymbol{\lambda}_{\mathrm{P}}^{n(k)}\end{bmatrix}$$

$$= -\begin{bmatrix}\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^{n(k)})^T\left(\mathbf{r}^n(\mathbf{x}^0(\boldsymbol{\mu}) + \boldsymbol{\Phi}\hat{\mathbf{x}}^{n(k)}) + \bar{\mathbf{C}}^T\boldsymbol{\lambda}_{\mathrm{P}}^{n(k)}\right) \\ \bar{\mathbf{C}}\mathbf{r}^n(\mathbf{x}^0(\boldsymbol{\mu}) + \boldsymbol{\Phi}\hat{\mathbf{x}}^{n(k)})\end{bmatrix}$$

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- How to handle infeasibility?
- How to solve?
- **Are the two methods ever equivalent?**
- How to apply hyper-reduction in a structure-preserving way?
- How do *a posteriori* error bounds compare with standard ROMs?

Are the two approaches ever equivalent?

Conservative Galerkin OΔE

$$\boldsymbol{\Phi}^T[\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_G^n) + \sum_{j=0}^{k}\alpha_j \bar{\mathbf{C}}^T \boldsymbol{\lambda}_G^{n-j}] = \mathbf{0}$$

$$\bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_G^n) = \mathbf{0}$$

Conservative LSPG OΔE

$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}_P^n)^T\left[\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_P^n) + \bar{\mathbf{C}}^T\boldsymbol{\lambda}_P^n\right] = \mathbf{0}$$

$$\bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_P^n) = \mathbf{0}$$

These are equivalent if, for some constant a,

$$\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n) = a\boldsymbol{\Phi} \quad \text{and} \quad \boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n)^T\bar{\mathbf{C}}^T\boldsymbol{\lambda}_P^n = a\sum_{j=0}^{k}\alpha_j\boldsymbol{\Phi}^T\bar{\mathbf{C}}^T\boldsymbol{\lambda}_G^{n-j}.$$

Recall $\boldsymbol{\Psi}^n(\hat{\mathbf{x}}^n) := (\alpha_0\mathbf{I} - \Delta t\beta_0\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\boldsymbol{\Phi}\hat{\mathbf{x}}^n;t))\,\boldsymbol{\Phi}$

Theorem: equivalence

The two approaches are equivalent (with $a = \alpha_0$)

1. in the limit of $\Delta t \to 0$, or
2. if the scheme is explicit $(\beta_0 = 0)$.

Further, the Lagrange multipliers are related as $\boldsymbol{\lambda}_P^n = \sum_{j=0}^{k}\alpha_j\boldsymbol{\lambda}_G^{n-j}$

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}} \; \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- How to handle infeasibility?
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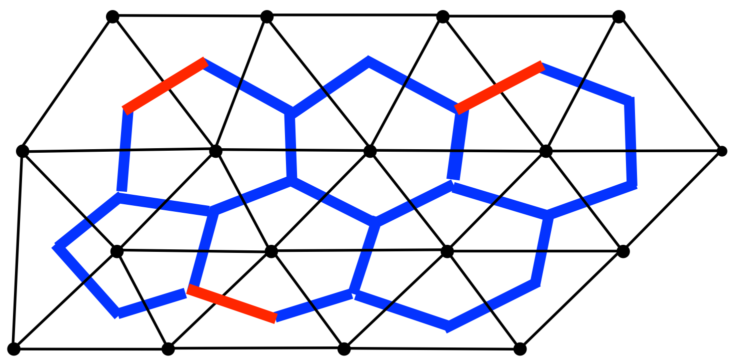
- **How to apply hyper-reduction in a structure-preserving way?**
- How do *a posteriori* error bounds compare with standard ROMs?

Hyper-reduction for finite-volume models

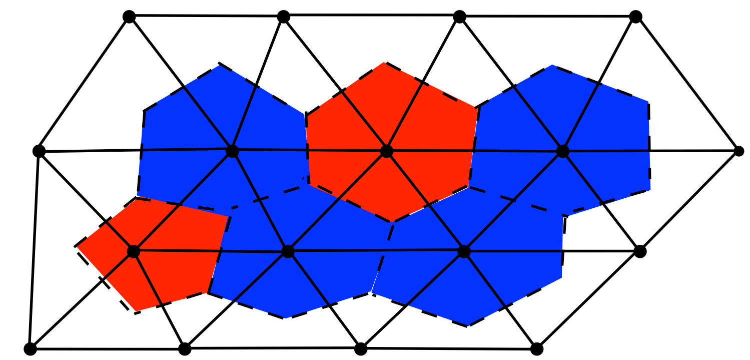
1. Residual gappy POD: $\tilde{\mathbf{r}} = \boldsymbol{\Phi}_{\mathrm{r}}(\mathbf{P}_{\mathrm{r}}\boldsymbol{\Phi}_{\mathrm{r}})^{+}\mathbf{P}_{\mathrm{r}}\mathbf{r}$, $\tilde{\mathbf{r}}^{n} = \boldsymbol{\Phi}_{\mathrm{r}}(\mathbf{P}_{\mathrm{r}}\boldsymbol{\Phi}_{\mathrm{r}})^{+}\mathbf{P}_{\mathrm{r}}\mathbf{r}^{n}$
2. Velocity gappy POD: $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{r}}^{n}$ computed from $\tilde{\mathbf{f}} = \boldsymbol{\Phi}_{\mathrm{f}}(\mathbf{P}_{\mathrm{f}}\boldsymbol{\Phi}_{\mathrm{f}})^{+}\mathbf{P}_{\mathrm{f}}\mathbf{f}$
3. Flux and source gappy POD

flux gappy POD



$$\tilde{\mathbf{h}} = \boldsymbol{\Phi}_{\mathrm{h}}(\mathbf{P}_{\mathrm{h}}\boldsymbol{\Phi}_{\mathrm{h}})^{+}\mathbf{P}_{\mathrm{h}}\mathbf{h}$$

source gappy POD



$$\tilde{\mathbf{f}}^{s} = \boldsymbol{\Phi}_{s}(\mathbf{P}_{s}\boldsymbol{\Phi}_{s})^{+}\mathbf{P}_{s}\mathbf{f}^{s}$$

- $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{r}}^{n}$ computed from $\tilde{\mathbf{f}} = \tilde{\mathbf{f}}^{g} + \tilde{\mathbf{f}}^{s}$ where $\tilde{\mathbf{f}}^{g} = \mathbf{B}\tilde{\mathbf{h}}$

+Structure preserving: approximated velocity is sum of flux and source

+ Less expensive: no need to compute all fluxes for a control volume

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^{p}}{\text{minimize}} \ \|\tilde{\mathbf{r}}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2 \quad \text{subject to } \bar{\mathbf{C}}\tilde{\tilde{\mathbf{r}}}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^{p}}{\text{minimize}} \ \|\tilde{\mathbf{r}}^{n}(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2 \quad \text{subject to } \bar{\mathbf{C}}\tilde{\tilde{\mathbf{r}}}^{n}(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

\+ Can apply different hyper-reduction to the objective $\tilde{\mathbf{r}}$ and constraints $\tilde{\tilde{\mathbf{r}}}$

\- Constraint hyper-reduction: no longer strictly conservative

\+ Constraint hyper-reduction: unneeded if no source and few subdomains

Questions

Conservative Galerkin

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}}\ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

Conservative LSPG

$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}}\ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to } \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$

- What are conditions for feasibility?
- How to handle infeasibility?
- How to solve?
- Are the two methods ever equivalent?
- How to apply hyper-reduction in a structure-preserving way?
- **How do *a posteriori* error bounds compare with standard ROMs?**

Discrete-time error bound: previous results

Theorem: state-space error bounds [C., Barone, Antil, 2017]

If the following conditions hold:

1. $\mathbf{f}(\cdot\,; t)$ is Lipschitz continuous with Lipschitz constant κ
2. The time step Δt is small enough such that $0 < h := |\alpha_0| - |\beta_0|\kappa\Delta t$,
3. A backward differentiation formula (BDF) time integrator is used,

$$\|\mathbf{x}^n - \mathbf{\Phi}\hat{\mathbf{x}}^n_{\mathrm{G}}\|_2 \leq \frac{1}{h}\|\mathbf{r}^n_{\mathrm{G}}(\mathbf{\Phi}\hat{\mathbf{x}}^n_{\mathrm{G}})\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \mathbf{\Phi}\hat{\mathbf{x}}^{n-\ell}_{\mathrm{G}}\|_2$$

$$\|\mathbf{x}^n - \mathbf{\Phi}\hat{\mathbf{x}}^n_{\mathrm{LSPG}}\|_2 \leq \frac{1}{h}\min_{\hat{\mathbf{v}}}\|\mathbf{r}^n_{\mathrm{LSPG}}(\mathbf{\Phi}\hat{\mathbf{v}})\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \mathbf{\Phi}\hat{\mathbf{x}}^{n-\ell}_{\mathrm{LSPG}}\|_2$$

+ LSPG sequentially minimizes the error bound

Discrete-time error bound: new results

Theorem: local state-space error bounds

If the following conditions hold:

1. $\mathbf{f}(\cdot;t)$ is Lipschitz continuous with Lipschitz constant κ
2. The time step Δt is small enough such that $0 < h := |\alpha_0| - |\beta_0|\kappa\Delta t$,
3. A backward differentiation formula (BDF) time integrator is used,

$$\|\mathbf{x}^n - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^n\|_2 \leq (1+\zeta_{\mathrm{G}})\frac{1}{h}\|\mathbf{r}_{\mathrm{G}}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^n)\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{G}}^{n-\ell}\|_2$$

$$\|\mathbf{x}^n - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^n\|_2 \leq \frac{1}{h}\|\mathbf{r}_{\mathrm{LSPG}}^n(\boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^n)\|_2 + \frac{1}{h}\sum_{\ell=1}^{k}|\alpha_\ell|\|\mathbf{x}^{n-\ell} - \boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^{n-\ell}\|_2$$
$$+ \frac{\zeta_{\mathrm{LSPG}}^n\Delta t}{h}\|(\mathbf{I} - [\mathbb{P}^n]^T\mathbb{P}^n)\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}_{\mathrm{LSPG}}^n)\|_2 + \frac{\zeta_{\mathrm{LSPG}}^n\|\Delta^n\|_2}{h^n}\sum_{\ell=0}^{k}|\alpha_\ell^n|\|\hat{\mathbf{x}}_{\mathrm{LSPG}}^{n-\ell}\|_2$$

- $\zeta_{\mathrm{G}} := \|\boldsymbol{\Sigma}_G^{-1}\mathbf{U}_G^T\bar{\mathbf{C}}\|_2$, $\zeta_{\mathrm{LSPG}} := \|[\boldsymbol{\Sigma}_{\mathrm{LSPG}}^n]^{-1}[\mathbf{U}_{\mathrm{LSPG}}^n]^T\bar{\mathbf{C}}\|_2$, $\Delta^n := \boldsymbol{\Psi}^n(\boldsymbol{\Phi}^T\boldsymbol{\Psi}^n)^{-1} - \boldsymbol{\Phi}$
- $\bar{\mathbf{C}}\boldsymbol{\Phi} = \mathbf{U}_{\mathrm{G}}\boldsymbol{\Sigma}_{\mathrm{G}}\mathbf{V}_{\mathrm{G}}^T$, $\bar{\mathbf{C}}\boldsymbol{\Psi}^n(\boldsymbol{\Phi}^T\boldsymbol{\Psi}^n)^{-1} = \mathbf{U}_{\mathrm{LSPG}}^n\boldsymbol{\Sigma}_{\mathrm{LSPG}}^n[\mathbf{V}_{\mathrm{LSPG}}^n]^T$

- State-space error bound is larger for both models
- LSPG no longer strictly minimizes the residual

Discrete-time error bound: new results

Lemma: local conserved-quantity error bounds

The error in the conserved quantities computed with either conservative Galerkin or conservative LSPG can be bounded as:

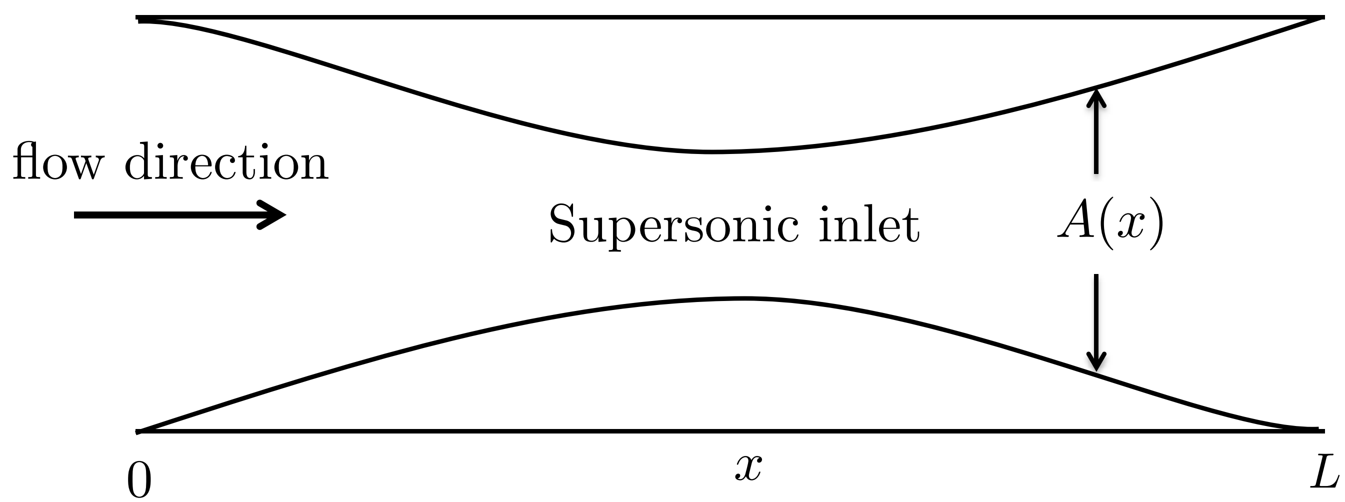
$$\|\bar{\mathbf{C}}(\mathbf{x}^n - \boldsymbol{\Phi}\hat{\mathbf{x}}^n)\|_2 \leq \sum_{\ell=0}^{k} \frac{|\beta_\ell^n|\Delta t}{|\alpha_0^n|}\|\bar{\mathbf{C}}\mathbf{f}(\mathbf{x}^{n-\ell}) - \bar{\mathbf{C}}\mathbf{f}(\boldsymbol{\Phi}\hat{\mathbf{x}}^{n-\ell})\|_2$$

$$+ \sum_{\ell=1}^{k} \frac{|\alpha_\ell^n|}{|\alpha_0^n|}\|\bar{\mathbf{C}}(\mathbf{x}^{n-\ell} - \boldsymbol{\Phi}\hat{\mathbf{x}}^{n-\ell})\|_2$$

- Error depends only on velocity error on *decomposed mesh*

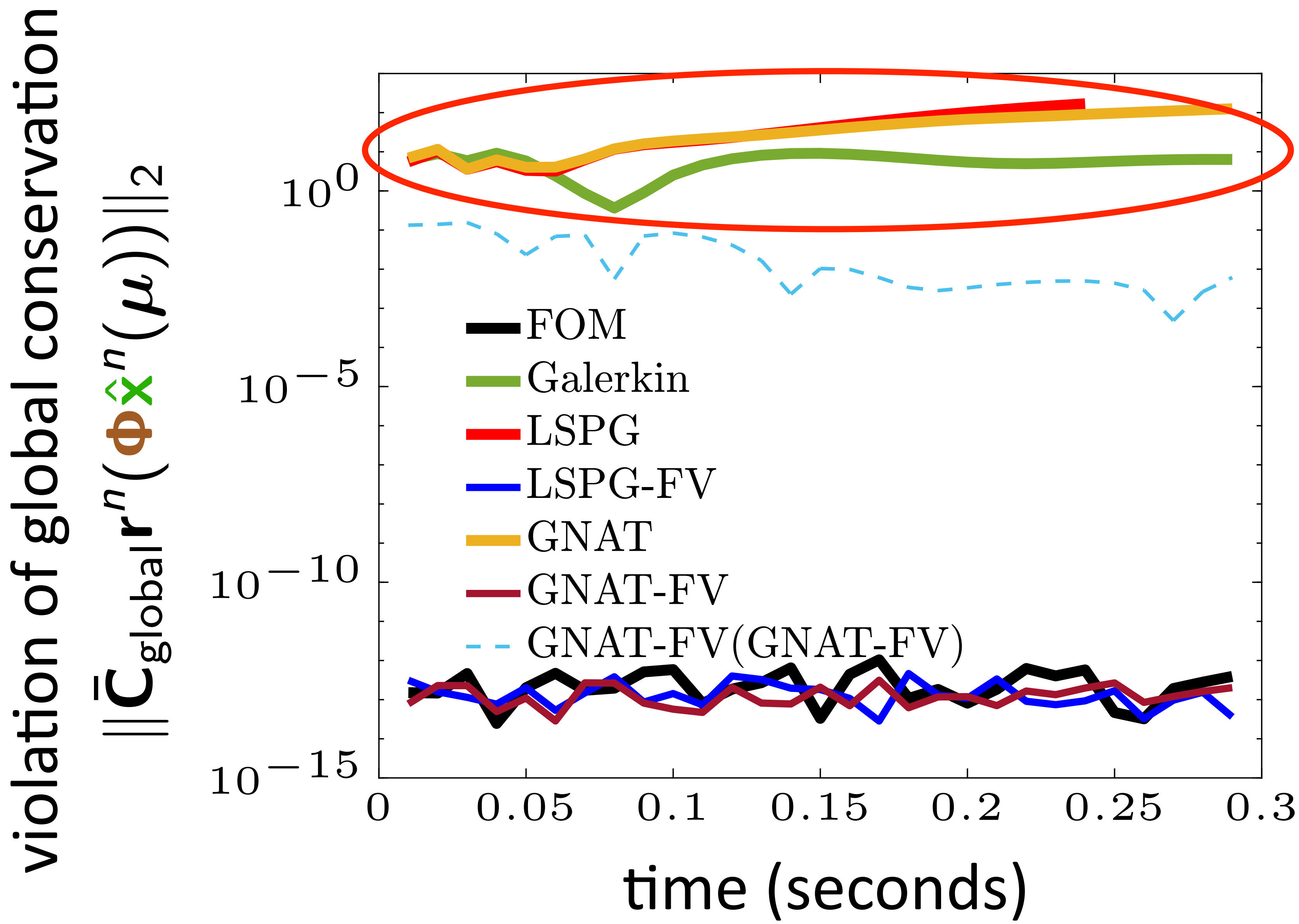
+ No source, global conservation: error due to flux error along boundary!

Quasi-1D Euler equation



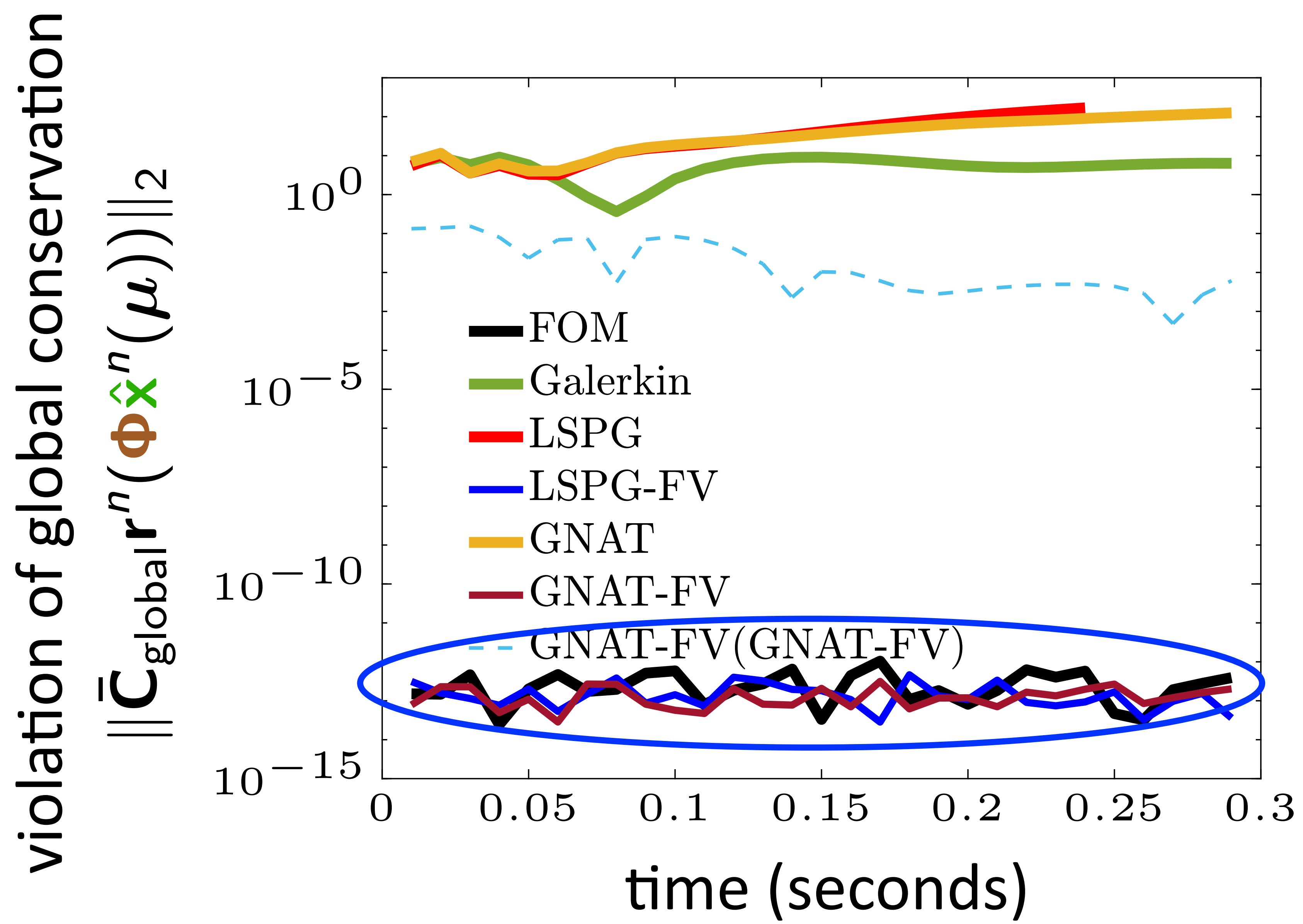
- 3 conserved variables: $u_1 = A\rho,\ u_2 = A\rho u,\ u_3 = Ae$
- Flux: $g_1 = A\rho u,\ g_2 = A(\rho u^2 + p),\ g_3 = A(e+p)u$
- Source: $s_1 = s_3 = 0,\ s_2 = p\frac{\partial A}{\partial x}$
- Domain length: $\mathrm{L}{=}0.25\ \mathrm{m}$
- Time domain: $t \in [0,\ 0.29\ \mathrm{s}]$
- Time integration: backward Euler with $\Delta t = 0.01\ \mathrm{s}$
- Parameter: the initial Mach number at the domain center
- Considered ROMs:
  - Galerkin
  - LSPG
  - LSPG-FV
  - GNAT: hyper-reduced objective
  - GNAT-FV: hyper-reduced objective
  - GNAT-FV(GNAT-FV): hyper-reduced objective & constraints

Global conservation $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



- Standard ROMs: significant global-conservation violation

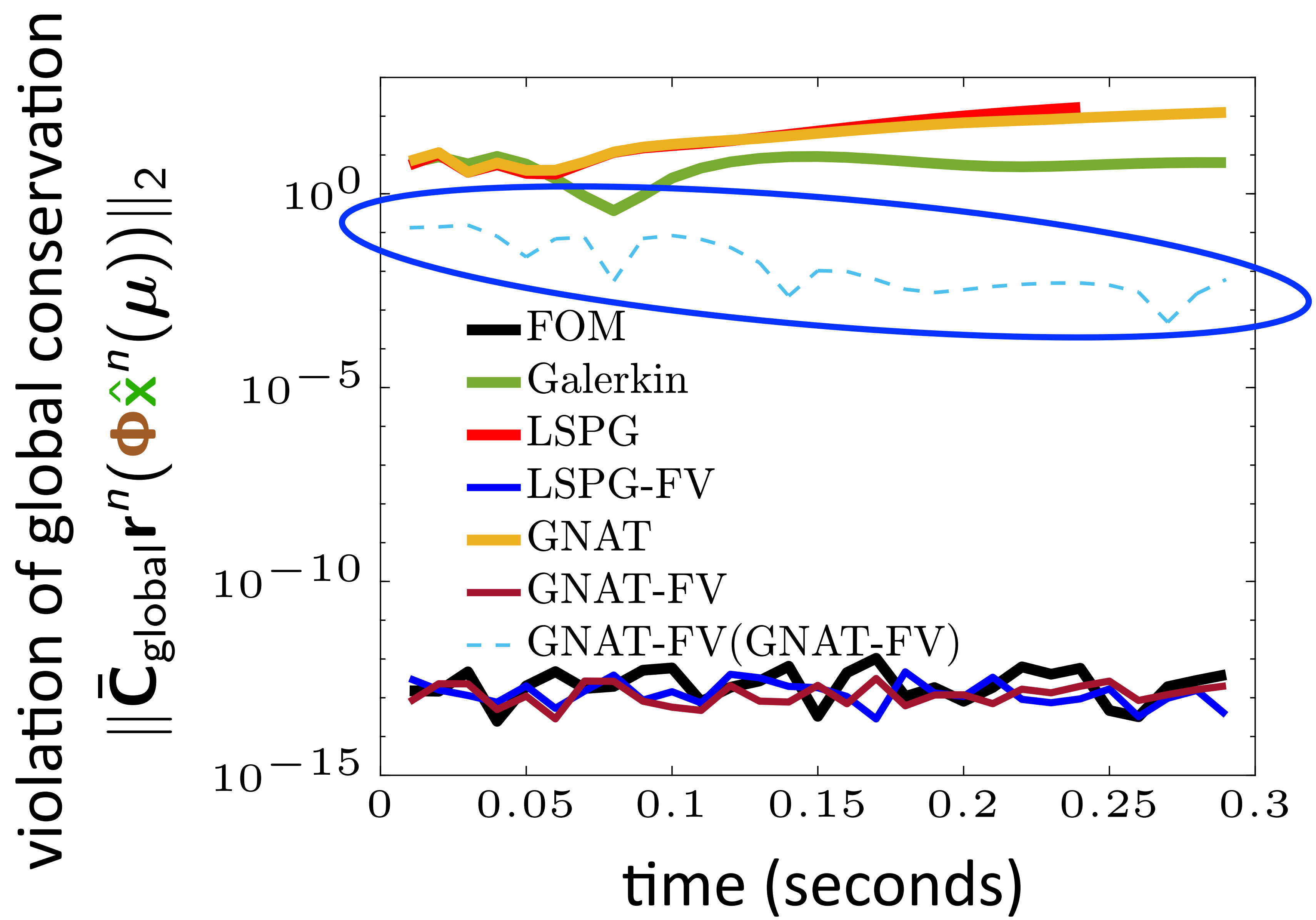
Global conservation $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



- Standard ROMs: significant global-conservation violation

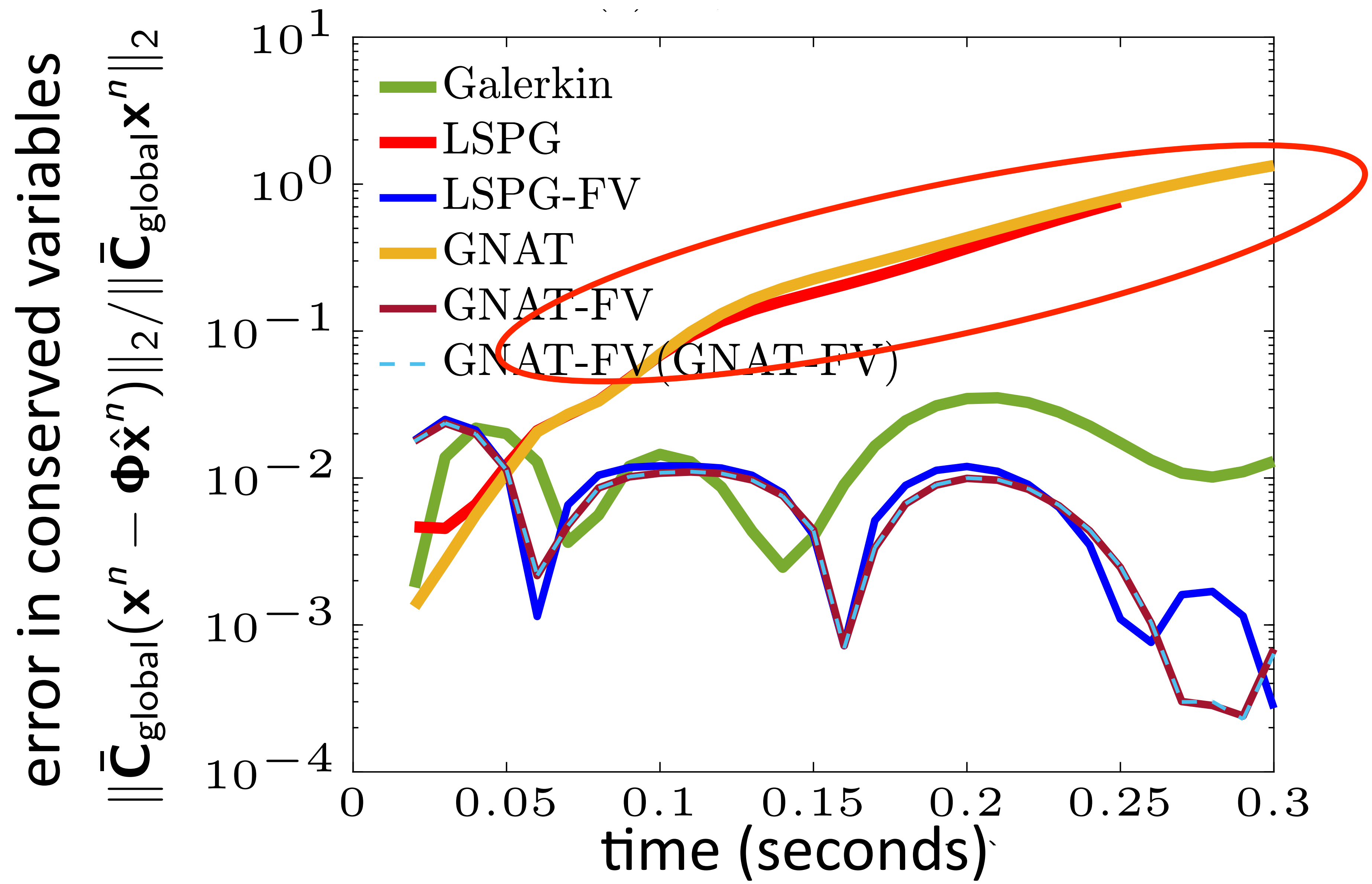
+ Conservative ROMs: global conservation satisfied (always feasible)

Global conservation $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



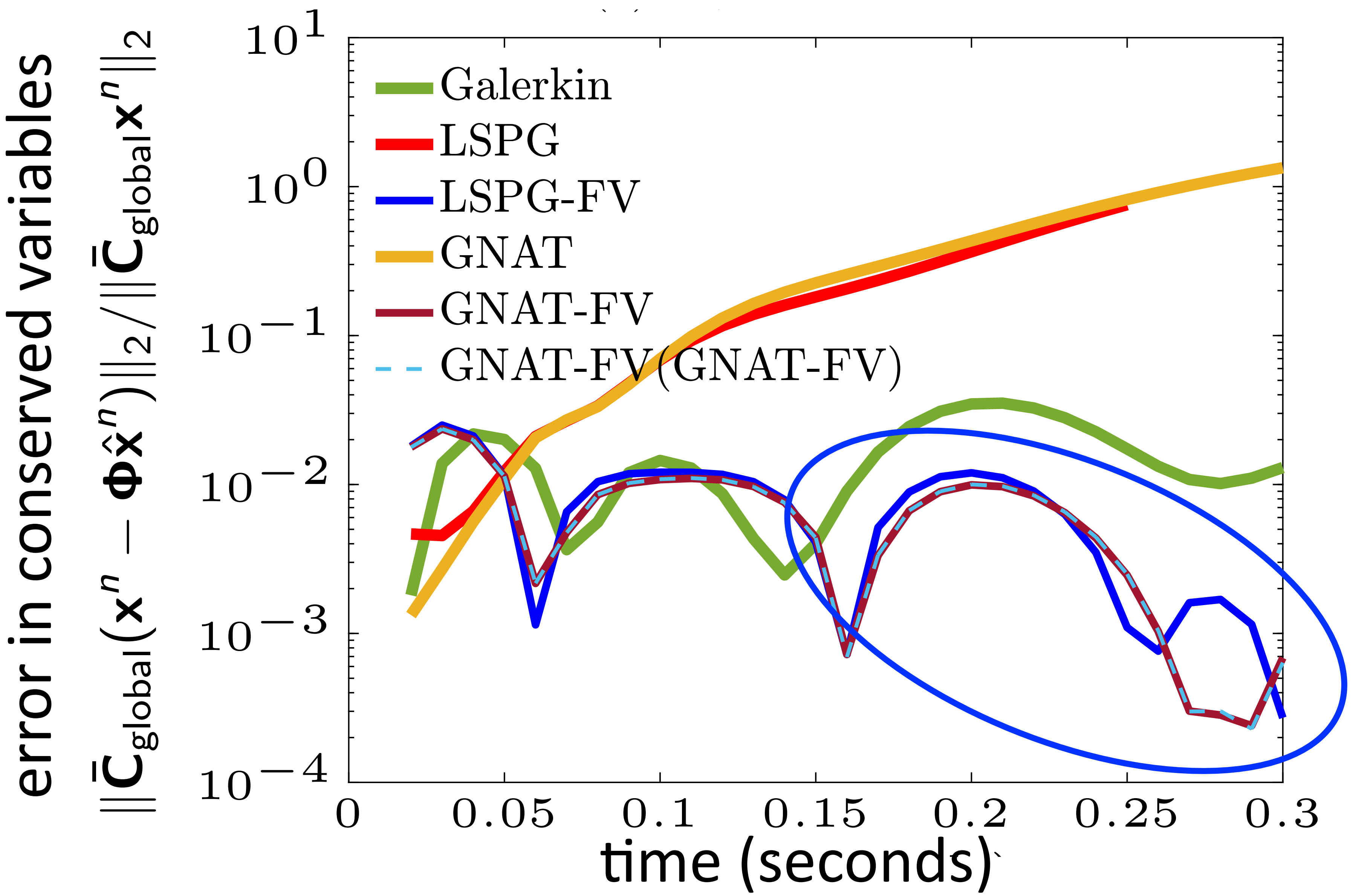
- Standard ROMs: significant global-conservation violation
+ Conservative ROMs: global conservation satisfied (always feasible)
+ Hyper-reduced constraints: relatively small global-conservation violation

Error in conserved variables $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



- Standard ROMs: can produce large errors in conserved quantities

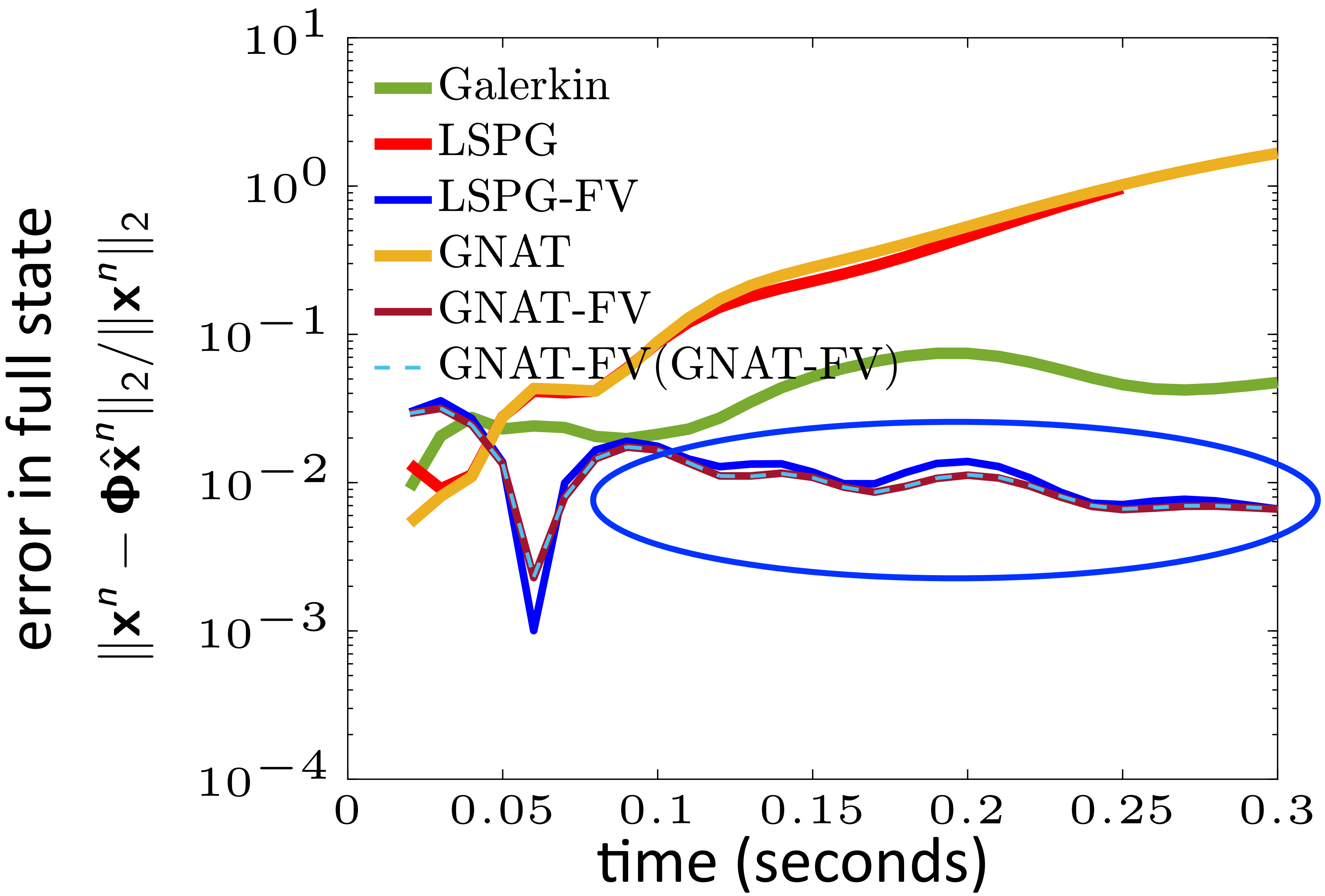
Error in conserved variables $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



- Standard ROMs: can produce large errors in conserved quantities

+ Conservative ROMs: small (but nonzero) errors in conserved quantities

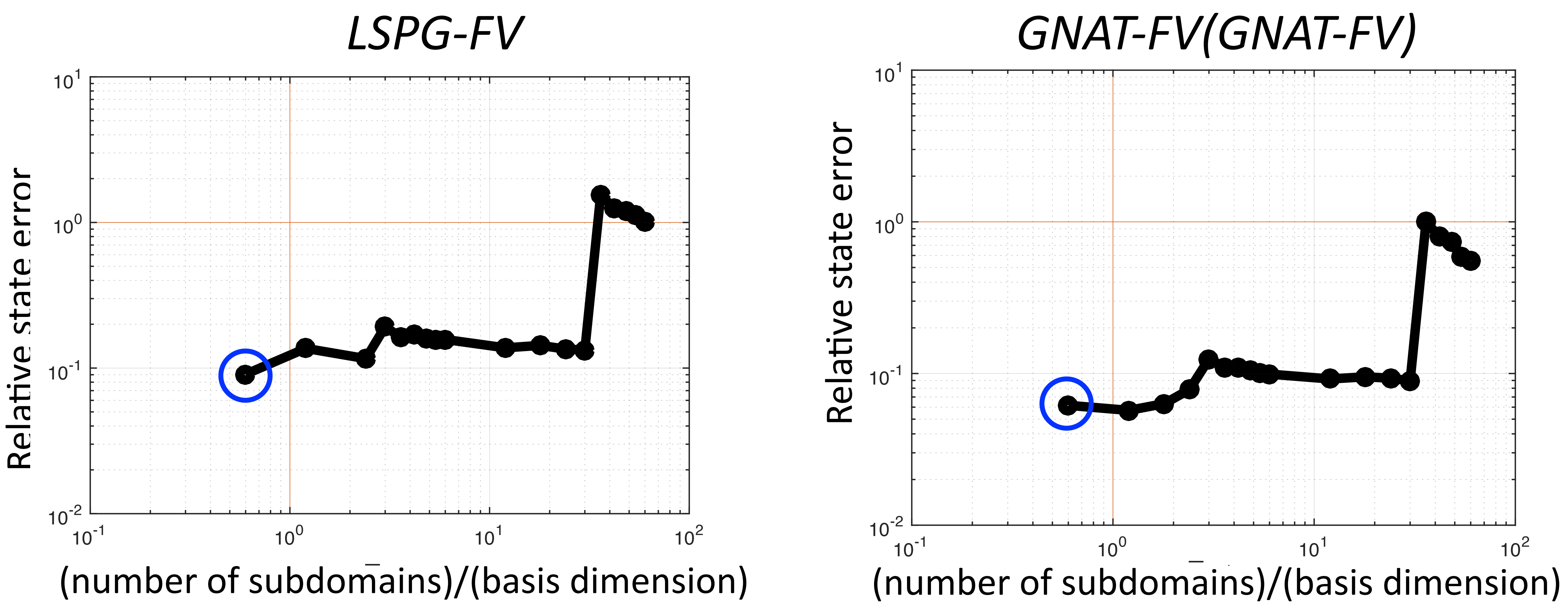
Error in conserved variables $(\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{global}})$



+ Conservative ROMs: smaller state-space errors
- Similar behavior of full-state error and globally-conserved quantity error!
+ Implies satisfying global conservation can improve overall accuracy

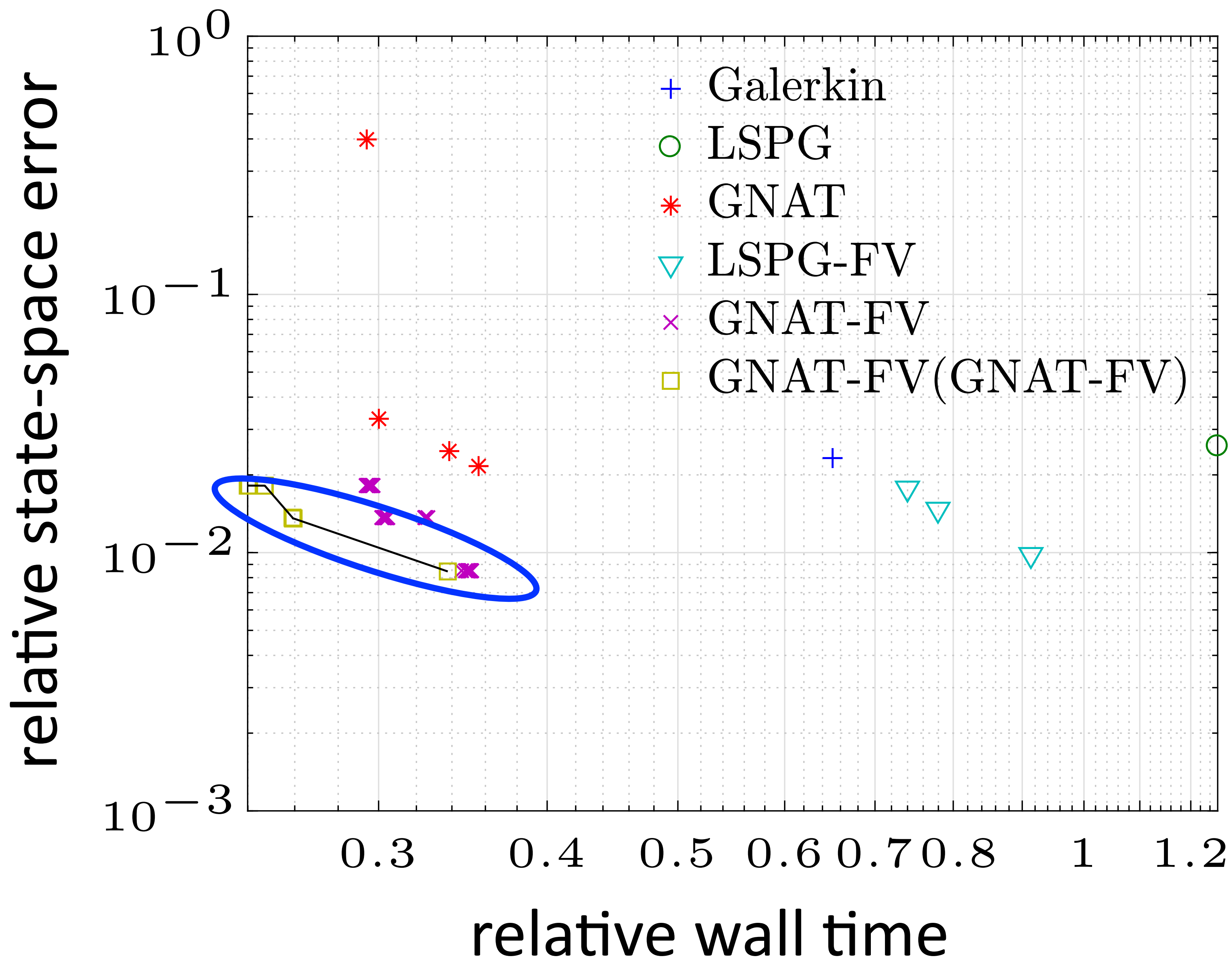
Varying number of subdomains

- If infeasible, adopt penalty formulation with $\rho = 10^3$



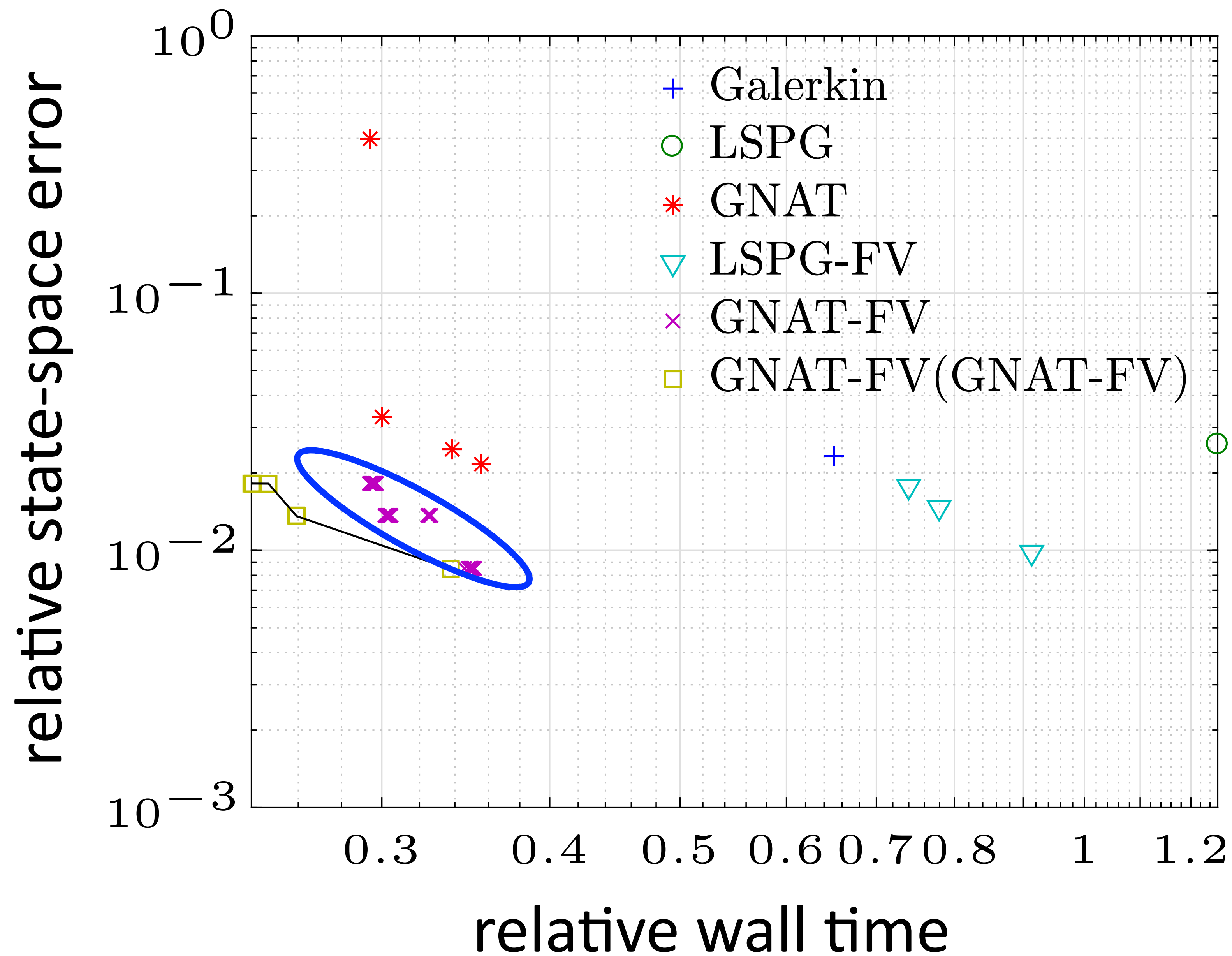
+ Global conservation yields the best performance
+ Global conservation reduces errors by 10X from the unconstrained case

Pareto optimality



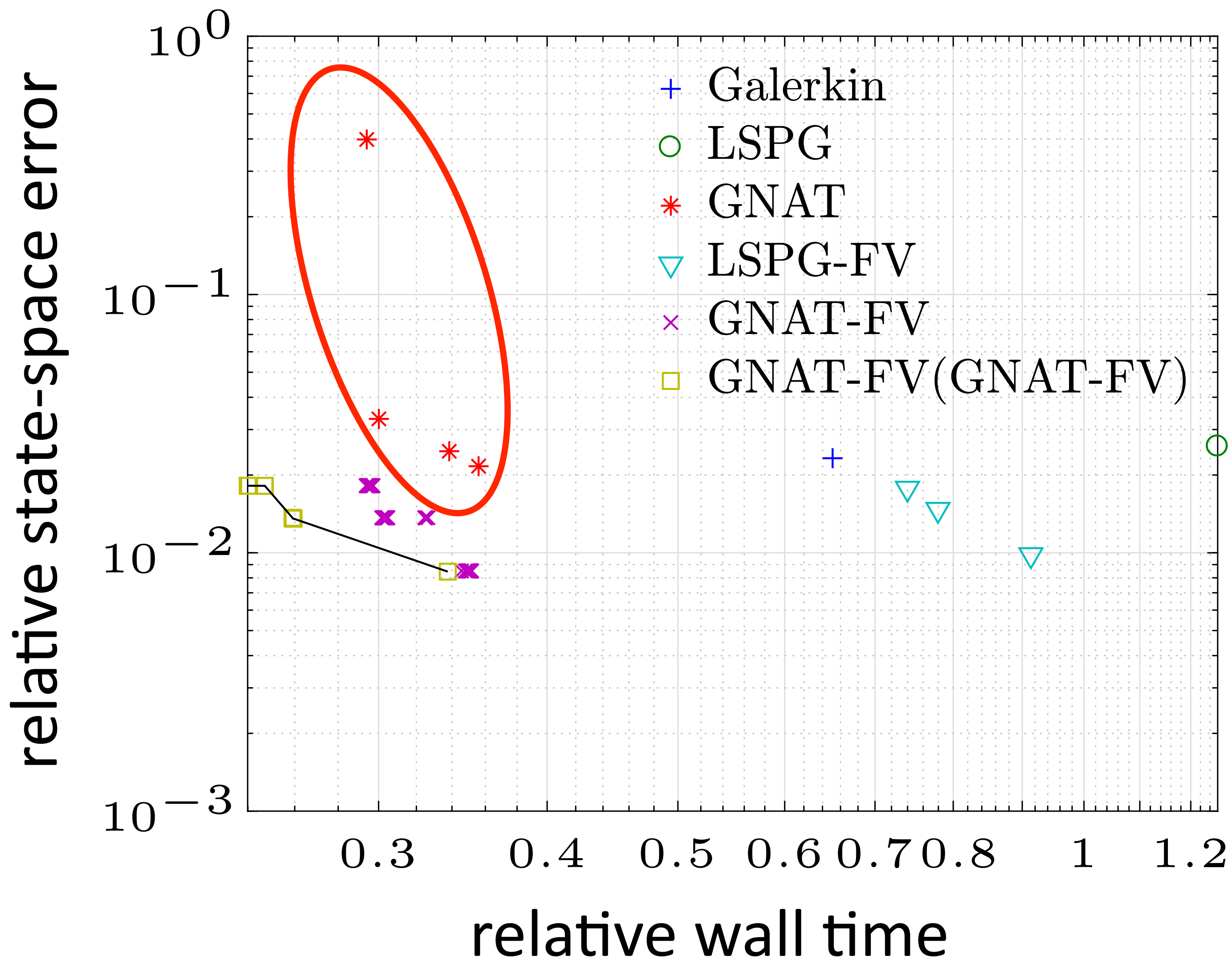
+ GNAT-FV(GNAT-FV) (hyper-reduced objective/constraints): Pareto optimal

Pareto optimality



- \+ GNAT-FV(GNAT-FV) (hyper-reduced objective/constraints): Pareto optimal
- \+ GNAT-FV (hyper-reduced objective, exact constraints): second-best

Pareto optimality



+ GNAT-FV(GNAT-FV) (hyper-reduced objective/constraints): Pareto optimal
+ GNAT-FV (hyper-reduced objective, exact constraints): second-best
- GNAT (hyper-reduced objective, no constraints): dominated

Conclusions

- Reduced-order models that enforce conservation
- Conditions that determine when conservation enforcement is ensured
- Ways to handle infeasibility
- Structure-preserving hyper-reduction that respects the velocity structure
- *A posteriori* error bounds
- Numerical experiments:
  - global conservation can reduce errors by 10X
  - hyper-reduced constraints nearly as accurate as strict constraints

Questions?

Reference: C., Choi, and Sargsyan. Conservative model reduction for finite-volume models. *Journal of Computational Physics*, 371:280–314, 2018.

Conservative Galerkin

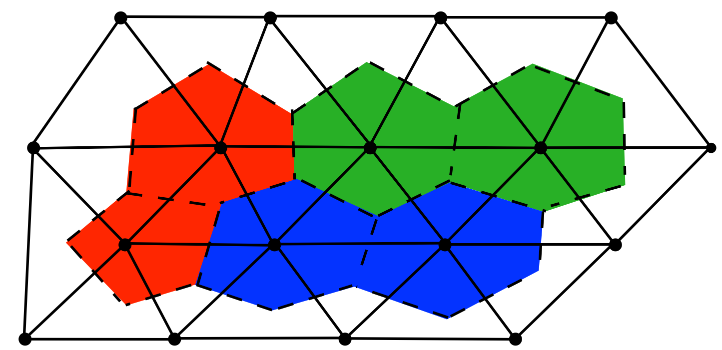
$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}}\ \|\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t)\|_2$$

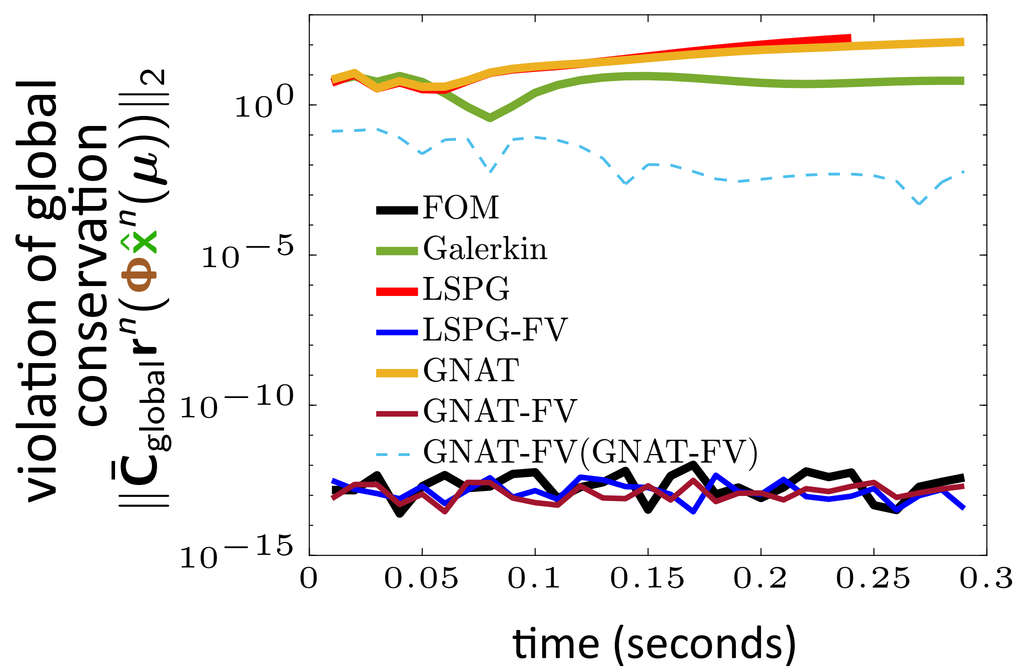
$$\text{subject to}\ \bar{\mathbf{C}}\mathbf{r}(\boldsymbol{\Phi}\hat{\mathbf{v}}, \boldsymbol{\Phi}\hat{\mathbf{x}}, t) = \mathbf{0}$$

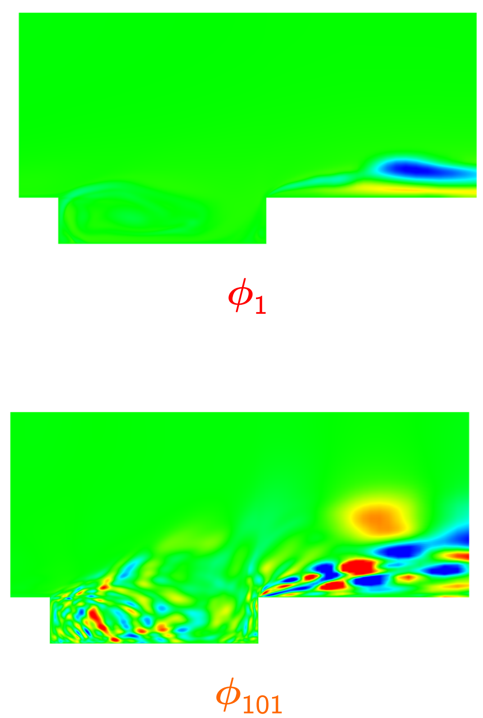
Conservative LSPG

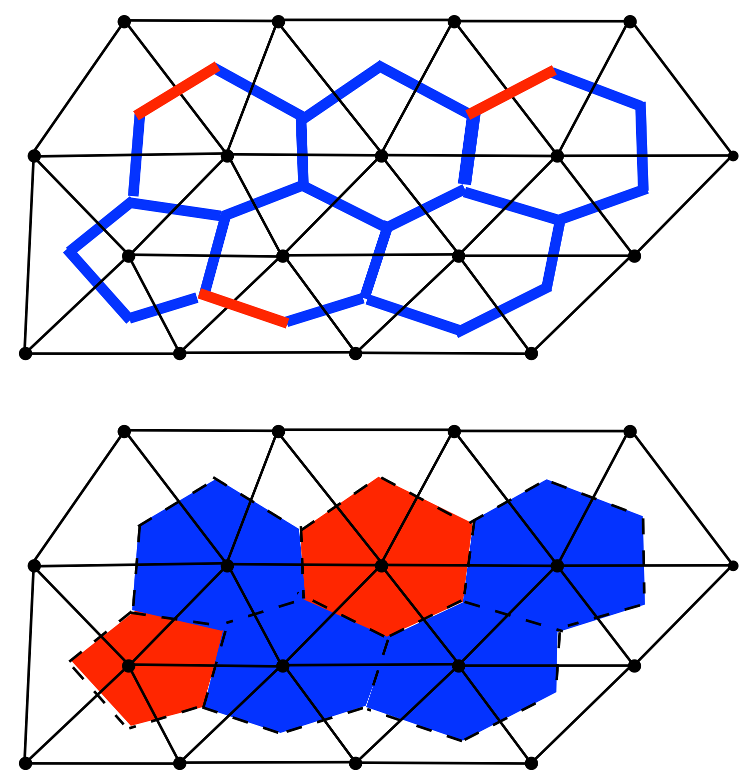
$$\underset{\hat{\mathbf{v}}\in\mathbb{R}^p}{\text{minimize}}\ \|\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}})\|_2$$

$$\text{subject to}\ \bar{\mathbf{C}}\mathbf{r}^n(\boldsymbol{\Phi}\hat{\mathbf{v}}) = \mathbf{0}$$









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